STOCHASTIC HYDROLOGY

Lecture -2
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Summary of the previous lecture

• Concept of a random variable
• Discrete and continuous random variables
• Probability mass function, density function and cumulative distribution functions
## Corrections in the slides of Lecture-1

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<th>Title of slide</th>
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<td>23</td>
<td>Continuous RVs</td>
<td>$f(x) = \lim_{dx \to \infty} \frac{P[x &lt; X \leq x + dx]}{dx}$</td>
<td>$f(x) = \lim_{dx \to 0} \frac{P[x &lt; X \leq x + dx]}{dx}$</td>
</tr>
<tr>
<td>32</td>
<td>Mixed distributions</td>
<td>$\int_{-\infty}^{d} f_1(x) + P[x = d] + \int_{d}^{\infty} f_2(x) = 1.0$</td>
<td>$\int_{-\infty}^{d} f_1(x) dx + P[x = d] + \int_{d}^{\infty} f_2(x) dx = 1.0$</td>
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<tr>
<td>39</td>
<td></td>
<td>$P[5 \leq x \leq 3]$</td>
<td>$P[5 \leq X \leq 3]$</td>
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Corrections in Lecture-1

Slide No.23-Continuous RVs

\[ f(x) = \lim_{dx \to \infty} \frac{P[x < X \leq x + dx]}{dx} \]

where \( \int_{-\infty}^{\infty} f(x)dx = 1 \)

CORRECTED AS

\[ f(x) = \lim_{dx \to 0} \frac{P[x < X \leq x + dx]}{dx} \]

where \( \int_{-\infty}^{\infty} f(x)dx = 1 \)
Bivariate Distributions

• In many situations, we would be interested in simultaneous behavior of two or more random variables. e.g., in hydrology, we may be interested in the joint behavior of
  - Rainfall – Runoff
  - Rainfall – Recharge
  - Rainfall intensity- Peak flood discharge
  - Temperature – Evaporation
  - Soil permeability – GW yield
  - Flow rates on two streams
Bi-variate Distributions

• We denote \((X,Y)\) as a two-dimensional random variable (or a two dimensional random vector).

• \(X\) and \(Y\) both discrete : two dimensional discrete r.v

• \(X\) and \(Y\) both continuous : two dimensional continuous r.v.

• It is possible that one of the rvs of \((X, Y)\), say, \(X\), is discrete while the other is continuous. In this course, however, we deal only with cases in which both \(X\) & \(Y\) are either discrete or continuous.
Probability distribution of \((X, Y)\)

- We define the \textit{joint probability mass function} of a two dimensional discrete r.v., \((X,Y)\) as,

\[
p(x_i, y_j) = P[X=x_i, Y=y_j]
\]

By this we imply, \(P[X = x_i, \text{AND } Y = y_j]\)

- \(p(x_i, y_j) \geq 0\)

- \(\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} p(x_i, y_j) = 1\)
Probability Distribution of \((X, Y)\)

- Similar to the CDF for one dimensional random variables, we define the \textbf{joint CDF} of the two dimensional discrete r.v., \((X,Y)\) as

\[
F(x, y) = \text{Prob}[X \leq x, Y \leq y]
\]

\[
= \sum_{x_i \leq x} \sum_{y_j \leq y} p(x_i, y_j)
\]

\[
F(\infty, \infty) = P[X \leq \infty, Y \leq \infty] = 1.0
\]
Example: Discrete two-d RV

Probability mass function

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
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<tbody>
<tr>
<td>y</td>
<td>0</td>
<td>0</td>
<td>0.04</td>
<td>0.05</td>
<td>0.07</td>
</tr>
<tr>
<td>1</td>
<td>0.03</td>
<td>0.04</td>
<td>0.06</td>
<td>0.07</td>
<td>0.08</td>
</tr>
<tr>
<td>2</td>
<td>0.02</td>
<td>0.05</td>
<td>0.05</td>
<td>0.07</td>
<td>0.05</td>
</tr>
<tr>
<td>3</td>
<td>0.01</td>
<td>0.03</td>
<td>0.05</td>
<td>0.07</td>
<td>0.07</td>
</tr>
</tbody>
</table>

\[
F(3,2) = \sum_{y=0}^{2} \sum_{x=0}^{3} p(x, y)
\]

\[
P[X \leq 3, Y \leq 2] = 0 + 0.04 + 0.05 + 0.07 + 0.03 + 0.04 + 0.06 + 0.07 + 0.02 + 0.05 + 0.05 + 0.07 + 0.01 + 0.03 + 0.05 + 0.07 = 0.55
\]
Joint pdf of (X, Y)

- For a continuous r.v. (X, Y), we define the joint probability density function, f(x, y), as
  1) $f(x, y) \geq 0$

$$2) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy = 1$$

This states that the total volume under the surface given by f(x, y) is 1.0

- The joint pdf, f(x, y) is not a probability.
- For small $\Delta x, \Delta y$ (+ve), $f(x, y) \Delta x \Delta y$ is approximately equal to $P[x \leq X \leq x + \Delta x, y \leq Y \leq y + \Delta y]$
Joint cdf of \((X, Y)\)

- The joint cumulative distribution function \(F(x, y)\) of the two dimensional random vector \((x, y)\) is defined as
  
  \[
  F(x, y) = P[X \leq x, Y \leq y]
  \]
  
  \[
  = \int_{-\infty}^{x} \int_{-\infty}^{y} f(x, y) \, dx \, dy
  \]

  It follows from the definition, that
  
  - \(F(\infty, \infty) = 1.0\)
  - \(F(-\infty, y) = F(x, -\infty) = 0\)
Flows in two adjacent streams are denoted as a random vector \((X, Y)\) with a joint pdf
\[
f(x, y) = c \quad \text{if } 5 \leq x \leq 10 ; 4 \leq y \leq 9
\]
\[
= 0, \quad \text{elsewhere}
\]

1. Obtain ‘c’
2. Obtain \(P[X > Y]\)
Example 1 (contd)

1. To determine ‘c’

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy = 1 \]

\[ \int_{4}^{5} \int_{9}^{10} c \, dx \, dy = 1 \]

\[ c \int_{4}^{5} \left[ x \right] \, dy = 1 \]

\[ 5c \left[ y \right]_{4}^{9} = 1 \]

\[ 25c = 1 \Rightarrow c = \frac{1}{25} \]
Example 1 (contd)

2. \( P[X \geq Y] = 1 - P[X \leq Y] \)

\[
= 1 - \int_{5}^{9} \int_{5}^{y} f(x, y) \, dx \, dy
\]

\[
= 1 - \frac{1}{25} \int_{5}^{9} \int_{5}^{y} dx \, dy
\]

\[
= 1 - \left\{ \frac{1}{25} \int_{5}^{9} (y - 5) \, dy \right\}
\]

\[
= 1 - \left\{ \frac{1}{25} \left[ \frac{y^2}{2} - 5y \right]_{5}^{9} \right\}
\]
Example 1 (contd)

\[ P[X > Y] = 0.68 \]

\[
1 - \left\{ \frac{1}{25} \left[ \frac{9^2}{2} - 5 \times 9 - \frac{5^2}{2} + 5 \times 5 \right] \right\} \\
= 1 - 0.32 \\
= 0.68
\]
Example 2

Consider the joint pdf

\[ f(x, y) = c(x^2+y^2) \quad 0 \leq x \leq 1 \]
\[ \quad \quad 0 \leq y \leq 1 \]
\[ = 0, \text{ elsewhere} \]

Obtain
1. the constant ‘c’
2. \( F(x, y) \)
3. \( P[X \leq 1/2, Y \leq 3/4] \)
4. \( P[X \geq Y] \)
5. \( P[X+Y \geq 1] \)
Example 2 (contd)

1. To obtain ‘c’, \( \iint_{-\infty}^{\infty} f(x, y) \, dx \, dy = 1 \)

\[
\int_{0}^{1} \int_{0}^{1} c(x^2 + y^2) \, dx \, dy = 1
\]

\[
\int_{0}^{1} c \left[ \frac{x^3}{3} + xy^2 \right]_{0}^{1} \, dy = 1
\]

\[
\int_{0}^{1} c \left[ \frac{1}{3} + y^2 \right] \, dy = 1
\]

\[
c \left[ \frac{y}{3} + 
\left. \frac{y^3}{3} \right|_{0}^{1} \right] = 1 \Rightarrow \frac{2c}{3} = 1 \Rightarrow c = \frac{3}{2}
\]
Example 2 (contd)

2. \( F(x, y) = \int_0^x \int_0^y f(x, y) \, dy \, dx = \int_0^x \int_0^y \frac{3}{2} \left( x^2 + y^2 \right) \, dy \, dx \)

\[
\begin{align*}
&= \int_0^x \left[ \frac{3}{2} \left( x^2 y + \frac{y^3}{3} \right) \right]_0^y \, dx \\
&= \int_0^x \frac{3}{2} \left( x^2 y + \frac{y^3}{3} \right) \, dx \\
&= \frac{3}{2} \left[ \frac{x^3 y}{3} + \frac{xy^3}{2} \right]_0^x \\
&= \frac{x^3 y}{2} + \frac{xy^3}{2} \\
\end{align*}
\]

\( F(x, y) = \frac{x^3 y + xy^3}{2} \quad 0 \leq x \leq 1 \quad 0 \leq y \leq 1 \)
Example 2 (contd)

3. \( P [X \leq 1/2, Y \leq 3/4] = \frac{\left(\frac{1}{2}\right)^3 \times \frac{3}{4} + \frac{1}{2} \times \left(\frac{3}{4}\right)^3}{2} \)

\[ = \frac{39}{256} \]

\[ = 0.152 \]
Example 2 (contd)

4. $P[Y \geq X]$

Limits $x \to 0$ to $y$
$y \to 0$ to $1$

$P[Y \geq X] = \int_0^1 \int_0^y \frac{3}{2} \left( x^2 + y^2 \right) dx dy$

$= \int_0^1 \left( \frac{x^3}{2} + \frac{3xy^2}{2} \right) \bigg|_0^y dy = \int_0^1 \left( \frac{1}{2} + \frac{3y^2}{2} - 2y^3 \right) dy$

$= \left[ \frac{y}{2} + \frac{y^3}{2} - \frac{y^4}{2} \right]_0^1 = \frac{1}{2}$
Example problem-2

5. \( P[X+Y \geq 1] \)

Limits \( x \to 0 \) to 1
\( y \to 1-x \) to 1

\[
P[X+Y \geq 1] = \int_{0}^{1} \int_{1-x}^{1} \frac{3}{2} (x^2 + y^2) \, dy \, dx
\]

\[
= \int_{0}^{1} \left( \frac{3x^2 y}{2} + \frac{y^3}{2} \right)_{1-x}^{1} \, dx
= \int_{0}^{1} \left( \frac{3x}{2} - \frac{3x^2}{2} + 2x^3 \right) \, dx
\]

\[
= \left[ \frac{3x^2}{4} - \frac{x^3}{2} + \frac{y^4}{2} \right]_{0}^{1} = \frac{3}{4}
\]
Marginal Probability Distribution

• We have seen \( f(x, y) \) as a joint probability distribution.
• In discrete case, \( p(x, y) = P[X = x, Y = y] \) indicates \( \text{prob } [X=x \text{ AND } Y=y] \).
• Consider the following distribution as in the previous numerical example.
### Marginal Probability Distribution

**Marginal distribution of Y**

<table>
<thead>
<tr>
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<th>x</th>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>Sum</th>
</tr>
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<tbody>
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<td></td>
</tr>
<tr>
<td>Sum</td>
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<td>0.21</td>
<td>0.28</td>
<td>0.29</td>
<td>1.00</td>
<td></td>
</tr>
</tbody>
</table>

**Marginal distribution of X**

- e.g., \( P[X \leq 3] = 0.06 + 0.16 + 0.21 + 0.28 = 0.71 \)
Marginal Probability Distribution

• An element in the body of the table indicates $P[X = x_i, Y = y_j]$.

• The marginal totals give $P[Y = y_j]$ and $P[X = x_i]$ respectively.

• For example, if we are interested in $P[Y = 0]$, this is given by marginal sum as 0.25.

• Since the event $P[Y = 0]$ can occur with $X=0$, $X=1$, ……… $X=5$. we have $P[Y=0, X=0 \text{ OR } Y=0, X=1 \text{ OR } …..]$

\[
P[Y = 0] = P[Y=0, X=0]+P[Y=0, X=1]+ P[Y=0, X=2]+……
\]

\[
\cdots \cdots\cdots\cdots\cdots\cdots\cdots\cdots P[Y=0, X=5]
\]

This indicates $P[Y=0]$ irrespective of the value of $X$
Marginal Probability Distribution

• In general, we may write as
  \[ p(x_i) = P[X=x_i] \]
  \[ = P[X=x_i, Y=y_1 \text{ or } X=x_i, Y=y_2 \text{ or } \ldots] \]
  \[ = \sum_{j=1}^{\infty} p(x_i, y_j) \]

• The function \( p(x_i) \) for \( i=1,2,\ldots \) is called the marginal distribution of \( X \).

• Analogously we define \( q(y_j) = \sum_{i=1}^{\infty} p(x_i, y_j) \) \( \forall j \), as the marginal distribution of \( Y \).
Marginal Density Functions

- In the continuous case, we proceed as follows
  - Let $f(x, y)$ denote the joint pdf of $(X, Y)$.
  - We define $g(x)$ and $h(y)$ as the marginal probability density functions of $X$ & $Y$ respectively as
    $$g(x) = \int_{-\infty}^{\infty} f(x, y)\,dy, \quad h(y) = \int_{-\infty}^{\infty} f(x, y)\,dx$$
- These marginal pdfs which are in fact derived from the joint pdf $f(x, y)$ correspond to the original pdf’s of the one-dimensional r.v.s $X$ and $Y$. 
Marginal Density Functions

This may be seen from
\[ P[c \leq X \leq d] = P[c \leq X \leq d, -\infty \leq Y \leq \infty] \]
\[
= \int_c^d \int_{-\infty}^{\infty} f(x, y) dy \, dx
\]
\[
= \int_c^d g(x) dx
\]

From the definitions of pdf’s, it is thus seen that \( g(x) \) is infact the original pdf of the r.v. \( X \)
Marginal Density Functions

Thus \( g(x) = \int_{-\infty}^{\infty} f(x, y) \, dy \) and \( F(x) = \int_{-\infty}^{\infty} g(x) \, dx \).

Similarly for the r.v. \( Y \)
That is, starting with the joint pdf \( f(x, y) \), we are able to get the pdfs of \( X \) & \( Y \) respectively.
For discrete case these results may be written as

\[
p[X = x_i] = \sum_{j=1}^{\infty} p(x_i, y_j) \quad \forall \quad i
\]

\[
p[Y = y_j] = \sum_{i=1}^{\infty} p(x_i, y_j) \quad \forall \quad j
\]
Example 3

Consider the joint pdf in the previous example

\[ f(x, y) = \begin{cases} 
1/25 & 5 \leq x \leq 10 \\
0 & 4 \leq y \leq 9 \\
0, & \text{elsewhere} 
\end{cases} \]

1. Obtain the marginal density \( g(x), h(y) \)
2. Obtain CDF \( G(x), H(y) \)
3. \( P[X \geq 7] \)
4. \( P[5 \leq Y \leq 8] \)
Example 3 (contd)

1. To obtain $g(x)$,

$$
g(x) = \int_{-\infty}^{\infty} f(x, y) dy \quad 5 \leq x \leq 10$$

$$
= \int_{4}^{9} \frac{1}{25} dy
$$

$$
= \left[ \frac{y}{25} \right]_{4}^{9} = \frac{1}{5}
$$

$$
g(x) = \frac{1}{5} \quad 5 \leq x \leq 10
$$
Example 3 (contd)

1. To obtain $h(y)$,

$$h(y) = \int_{-\infty}^{\infty} f(x, y) \, dx \quad 4 \leq y \leq 9$$

$$= \int_{5}^{10} \frac{1}{25} \, dx$$

$$= \left[ \frac{x}{25} \right]_{5}^{10} = \frac{1}{5}$$

$$h(y) = \frac{1}{5} \quad 4 \leq y \leq 9$$
Example 3 (contd)

2. To obtain \( G(x) \),

\[
G(x) = \int_{-\infty}^{x} g(x) \, dx \quad 5 \leq x \leq 10
\]

\[
= \int_{5}^{x} \frac{1}{5} \, dx
\]

\[
= \left[ \frac{x}{5} \right]^{x}_{5}
\]

\[
G(x) = \frac{x - 5}{5} \quad 5 \leq x \leq 10
\]
Example 3 (contd)

2. To obtain \( H(y) \),

\[
H(y) = \int_{-\infty}^{y} h(y) \, dy \quad 4 \leq y \leq 9
\]

\[
= \int_{4}^{y} \frac{1}{5} \, dy
\]

\[
= \left[ \frac{y}{5} \right]^{y}_{4}
\]

\[
H(y) = \frac{y - 4}{5} \quad 4 \leq y \leq 9
\]
Example 3 (contd)

3. \( P[X \geq 7] = 1 - P[X \leq 7] \)
   \[ = 1 - G(7) \]
   \[ = 1 - \frac{7 - 5}{5} = \frac{3}{5} \]

4. \( P[5 \leq Y \leq 8] = H(8) - H(5) \)
   \[ = \frac{4}{5} - \frac{1}{5} = \frac{3}{5} \]
Example 4

Consider the joint pdf

\[ f(x, y) = e^{-y} \quad x > 0 \]
\[ y \geq x \]

1. Obtain the marginal density \( g(x) \)
2. \( P[X \geq 2] \)
Example 4 (contd)

1. To obtain \( g(x) \),

\[
g(x) = \int_{-\infty}^{\infty} f(x, y) dy \quad x > 0
\]

\[
= \int_{x}^{\infty} e^{-y} dy
\]

\[
= \left[ -e^{-y} \right]_{x}^{\infty} = e^{-x}
\]

\[
G(x) = \int_{0}^{x} e^{-x} dx = \left[ -e^{-x} \right]_{0}^{x}
\]

\[
= 1 - e^{-x} \quad x > 0
\]
Example 4 (contd)

2. \[ P[X \geq 2] = 1 - P[X \leq 2] \]

\[ = 1 - (1 - e^{-2}) \]

\[ = e^{-2} \]
Conditional Distribution

• A marginal distribution is the distribution of one variable regardless of the value of the second variable.
• A joint distribution is the simultaneous occurrence of the given values of the two variables.
• The distribution of one variable with conditions placed on the second variable is called conditional distribution. For example,
  – Distribution of X, given that \( Y=y_0 \) or distribution of Y given that \( c \leq X \leq d \) etc.
Conditional Distribution

**Definition:** (X, Y) is a continuous two dimensional r.v. with a joint pdf of f(x, y).

- Let g(x) and h(y) be the marginal pdfs of X and Y respectively
- The conditional pdf of X given Y = y is defined as

\[
g(x/y) = \frac{f(x, y)}{h(y)} \quad h(y) > 0
\]

Read as x given y
Conditional Distribution

• The conditional pdf of $Y$ given $X = x$ is defined as

$$h(y/x) = \frac{f(x,y)}{g(x)} \quad g(x) > 0$$

• The conditional pdf of $X$ given $Y \in \mathbb{R}$ and conditional pdf of $Y$ given $X \in \mathbb{R}$ is defined as

$$g(x/y \in R) = \frac{\int_{R} f(x,y) dy}{\int_{R} h(y) dy}, \quad h(y/x \in R) = \frac{\int_{R} f(x,y) dx}{\int_{R} g(x) dx}$$
Conditional Distribution

The conditional pdfs $g(x/y)$ and $h(y/x)$ satisfy all conditions for a pdf.

For a given $y$, $g(x/y)>0$, as both $f(x,y)$ and $h(y)$ are positive.

$$\int_{-\infty}^{\infty} g(x/y) = \int_{-\infty}^{\infty} \frac{f(x,y)}{h(y)} \, dx$$

$$= \frac{1}{h(y)} \int_{-\infty}^{\infty} f(x,y) \, dx$$

$$= \frac{h(y)}{h(y)} = 1.0$$

Cumulative conditional distributions

$$G(x/y) = \int_{-\infty}^{x} g(x/y) \, dx, \quad H(y/x) = \int_{-\infty}^{y} h(y/x) \, dy$$
Example 5

Consider the joint pdf

\[ f(x, y) = \frac{x(1 + 3y^2)}{4} \quad 0 \leq x \leq 2 \quad 0 \leq y \leq 1 \]

= 0, elsewhere

1. Obtain \( h(y/x) \)

2. \( P[\frac{1}{2} \leq Y \leq 1 | X = 1] \)

3. \( P[Y \leq \frac{3}{4} | X \leq 1] \)
Example 5 (contd)

1. To obtain $h(y/x)$, $g(x)$ is first obtained

$$g(x) = \int_{-\infty}^{\infty} f(x, y) dy \quad 0 \leq x \leq 2$$

$$= \int_{0}^{1} x \left( \frac{1 + 3y^2}{4} \right) dy$$

$$= \frac{1}{4} \left[ xy + y^3 \right]_{0}^{1}$$

$$g(x) = \frac{(x + 1)}{4} \quad 0 \leq x \leq 2$$
Example 5 (contd)

\[ h(y/x) = \frac{f(x, y)}{g(x)} = \frac{x(1 + 3y^2)}{x + 1} \]

2. \[ P[\frac{1}{2} \leq Y \leq \frac{1}{X}=x] = \int_{\frac{1}{2}}^{1} \frac{x}{x + 1} \left( 1 + 3y^2 \right) dy \]

\[ = \frac{x}{x + 1} \left[ y + y^3 \right]_{\frac{1}{2}}^{1} \]

\[ = \frac{11}{8} \left( \frac{x}{x + 1} \right) \]
Example 5 (contd)

2. \( P[1/2 \leq Y \leq 1/X=1] = \frac{11}{8} \left( \frac{1}{1+1} \right) = \frac{11}{16} \)

3. To obtain \( P[Y \leq 3/4/X \leq 1] \),

\[
h(y/x \in R) = \frac{\int f(x, y) \, dx}{\int g(x) \, dx}
\]

\[
\Rightarrow h(y/x \leq 1) = \frac{\int_{0}^{1} f(x, y) \, dx}{\int_{0}^{1} g(x) \, dx}
\]
Example 5 (contd)

\[ \int_{0}^{1} f(x, y)dx = \int_{0}^{1} \frac{x(1 + 3y^2)}{4} \, dx \]

\[ = \frac{1 + 3y^2}{4} \left[ \frac{x^2}{2} \right]_{0}^{1} \]

\[ = \frac{1 + 3y^2}{8} \]

\[ \int_{0}^{1} g(x)dx = \int_{0}^{1} \frac{(x + 1)}{4} \, dx \]

\[ = \frac{1}{4} \left[ \frac{x^2}{2} + x \right]_{0}^{1} = \frac{3}{8} \]
Example 5 (contd)

\[ h(y/x \leq 1) = \frac{1+3y^2}{3} \]

\[ h(y \leq 3/4/x \leq 1) = \int_0^{\frac{3}{4}} \frac{1+3y^2}{3} dy \]

\[ = \frac{1}{3} \left[ y + y^3 \right]_0^{\frac{3}{4}} \]

\[ = \frac{1}{3} \left[ \frac{3}{4} + \frac{27}{64} \right] = \frac{75}{192} \]
Independence of two random variables

- Intuitively, X and Y are independent r.v.s if the distribution of one r.v. does not influence distribution of the other r.v.
- Independence is a very useful assumption for hydrologic analysis in many situations. However, physically the assumption must have a sound basis.
• As an example, inflow to a reservoir (X) and the rainfall in the command area (Y) may be taken as independent, if the command area is far removed from the reservoir.
In water quality problems, for example, pollutant load (X) and stream flow (Y) may be treated as independent variables.
Independent R.V.

- When two r.v.s are independent, $g(x/y)=g(x)$
  - Distribution of $x$ given $y$ is independent of $y$ and hence the original pdf itself gives the conditional pdf

$$
g(x/y) = \frac{f(x, y)}{h(y)}$$

$$
g(x) = \frac{f(x, y)}{h(y)}$$

$$
f(x, y) = g(x).h(y)$$
Independent R.V.

• The random variables $X$ and $Y$ are stochastically independent if and only if their joint density is equal to the product of their marginal densities.
• For discrete case, the two r.v.s are independent if and only if
  $$p(x_i, y_j) = p(x_i) \cdot p(y_j) \quad \forall \ i,j$$