

# Module

# 4

## Analysis of Statically Indeterminate Structures by the Direct Stiffness Method

# Lesson

# 23

# The Direct Stiffness Method: An Introduction

## Instructional Objectives:

After reading this chapter the student will be able to

1. Differentiate between the direct stiffness method and the displacement method.
2. Formulate flexibility matrix of member.
3. Define stiffness matrix.
4. Construct stiffness matrix of a member.
5. Analyse simple structures by the direct stiffness matrix.

## 23.1 Introduction

All known methods of structural analysis are classified into two distinct groups:-

- (i) force method of analysis and
- (ii) displacement method of analysis.

In module 2, the force method of analysis or the method of consistent deformation is discussed. An introduction to the displacement method of analysis is given in module 3, where in slope-deflection method and moment- distribution method are discussed. In this module the direct stiffness method is discussed. In the displacement method of analysis the equilibrium equations are written by expressing the unknown joint displacements in terms of loads by using load-displacement relations. The unknown joint displacements (the degrees of freedom of the structure) are calculated by solving equilibrium equations. The slope-deflection and moment-distribution methods were extensively used before the high speed computing era. After the revolution in computer industry, only direct stiffness method is used.

The displacement method follows essentially the same steps for both statically determinate and indeterminate structures. In displacement /stiffness method of analysis, once the structural model is defined, the unknowns (joint rotations and translations) are automatically chosen unlike the force method of analysis. Hence, displacement method of analysis is preferred to computer implementation. The method follows a rather a set procedure. The direct stiffness method is closely related to slope-deflection equations.

The general method of analyzing indeterminate structures by displacement method may be traced to Navier (1785-1836). For example consider a four member truss as shown in Fig.23.1. The given truss is statically indeterminate to second degree as there are four bar forces but we have only two equations of equilibrium. Denote each member by a number, for example (1), (2), (3) and (4). Let  $\alpha_i$  be the angle, the  $i$ -th member makes with the horizontal. Under the

action of external loads  $P_x$  and  $P_y$ , the joint  $E$  displaces to  $E'$ . Let  $u$  and  $v$  be its vertical and horizontal displacements. Navier solved this problem as follows.

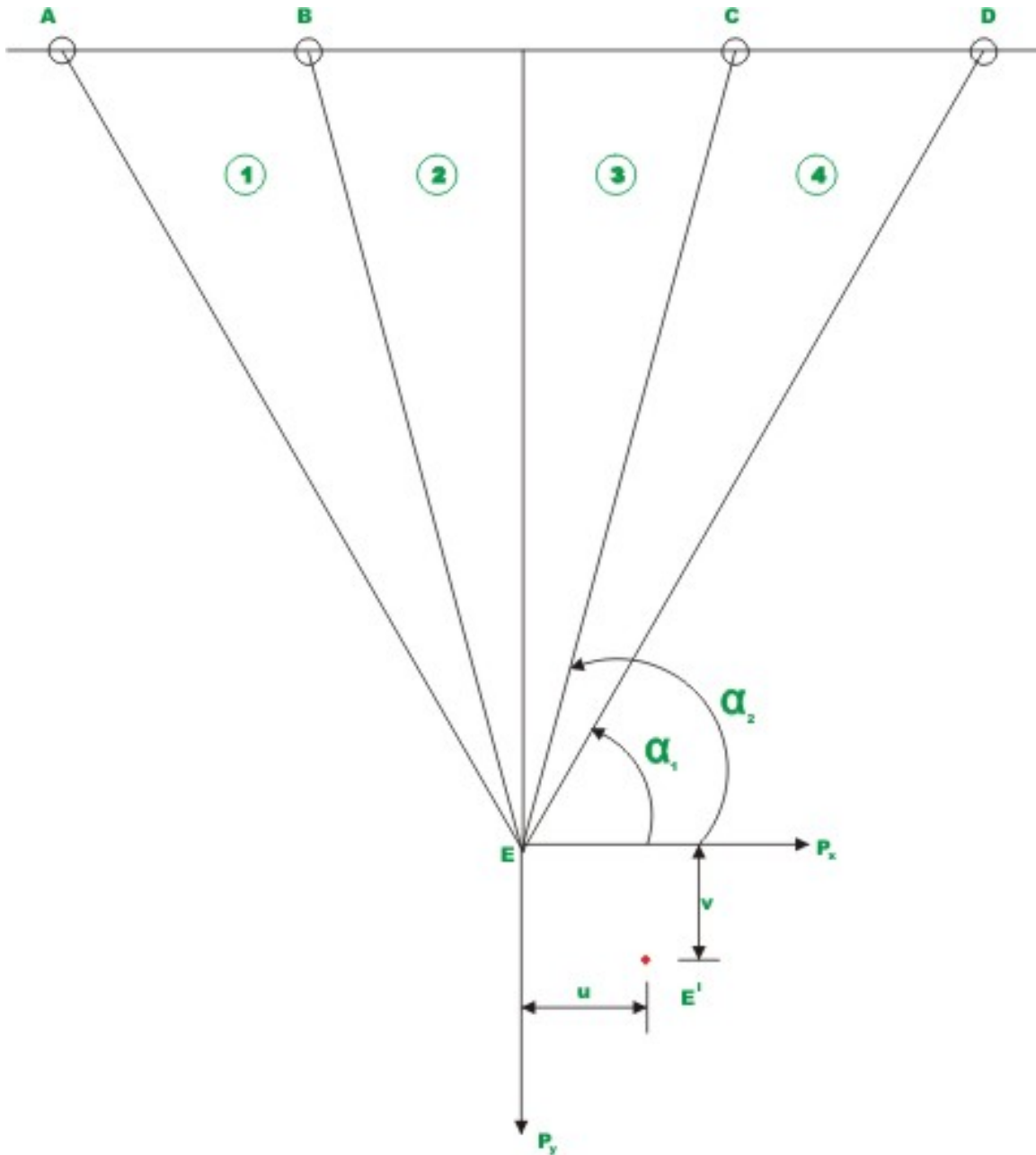
In the displacement method of analysis  $u$  and  $v$  are the only two unknowns for this structure. The elongation of individual truss members can be expressed in terms of these two unknown joint displacements. Next, calculate bar forces in the members by using force–displacement relation. Now at  $E$ , two equilibrium equations can be written viz.,  $\sum F_x = 0$  and  $\sum F_y = 0$  by summing all forces in  $x$  and  $y$  directions. The unknown displacements may be calculated by solving the equilibrium equations. In displacement method of analysis, there will be exactly as many equilibrium equations as there are unknowns.

Let an elastic body is acted by a force  $F$  and the corresponding displacement be  $u$  in the direction of force. In module 1, we have discussed force- displacement relationship. The force ( $F$ ) is related to the displacement ( $u$ ) for the linear elastic material by the relation

$$F = ku \quad (23.1)$$

where the constant of proportionality  $k$  is defined as the stiffness of the structure and it has units of force per unit elongation. The above equation may also be written as

$$u = aF \quad (23.2)$$



**Fig. 23.1 Four member truss**

The constant  $a$  is known as flexibility of the structure and it has a unit of displacement per unit force. In general the structures are subjected to  $n$  forces at  $n$  different locations on the structure. In such a case, to relate displacement at  $i$  to load at  $j$ , it is required to use flexibility coefficients with subscripts. Thus the flexibility coefficient  $a_{ij}$  is the deflection at  $i$  due to unit value of force applied at  $j$ . Similarly the stiffness coefficient  $k_{ij}$  is defined as the force generated at  $i$

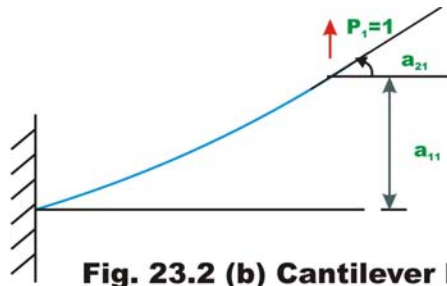
due to unit displacement at  $j$  with all other displacements kept at zero. To illustrate this definition, consider a cantilever beam which is loaded as shown in Fig.23.2. The two degrees of freedom for this problem are vertical displacement at  $B$  and rotation at  $B$ . Let them be denoted by  $u_1$  and  $u_2 (= \theta_1)$ . Denote the vertical force  $P$  by  $P_1$  and the tip moment  $M$  by  $P_2$ . Now apply a unit vertical force along  $P_1$  and calculate deflection  $u_1$  and  $u_2$ . The vertical deflection is denoted by flexibility coefficient  $a_{11}$  and rotation is denoted by flexibility coefficient  $a_{21}$ . Similarly, by applying a unit force along  $P_2$ , one could calculate flexibility coefficient  $a_{12}$  and  $a_{22}$ . Thus  $a_{12}$  is the deflection at 1 corresponding to  $P_2$  due to unit force applied at 2 in the direction of  $P_2$ . By using the principle of superposition, the displacements  $u_1$  and  $u_2$  are expressed as the sum of displacements due to loads  $P_1$  and  $P_2$  acting separately on the beam. Thus,

$$\begin{aligned} u_1 &= a_{11}P_1 + a_{12}P_2 \\ u_2 &= a_{21}P_1 + a_{22}P_2 \end{aligned} \quad (23.3a)$$

The above equation may be written in matrix notation as

$$\{u\} = [a]\{P\}$$

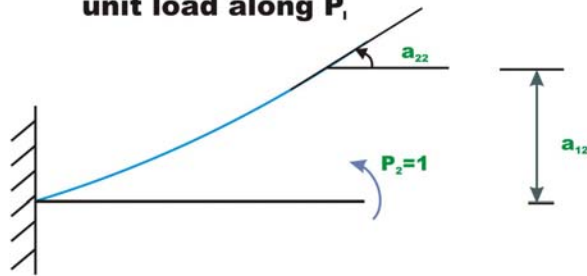
where,  $\{u\} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$ ;  $\{a\} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ ; and  $\{P\} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$



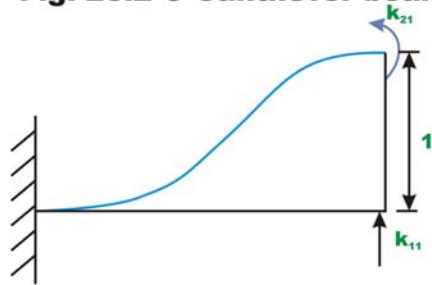
**Fig. 23.2 (b) Cantilever beam with unit load along  $P_1$**



**Fig.23.2(a) Cantilever beam**



**Fig. 23.2 © Cantilever beam with unit moment along  $P_2$**



**Fig. 23.2(d) Cantilever beam with unit displacement along  $u_1$**

The forces can also be related to displacements using stiffness coefficients. Apply a unit displacement along  $u_1$  (see Fig.23.2d) keeping displacement  $u_2$  as zero. Calculate the required forces for this case as  $k_{11}$  and  $k_{21}$ . Here,  $k_{21}$  represents force developed along  $P_2$  when a unit displacement along  $u_1$  is introduced keeping  $u_2 = 0$ . Apply a unit rotation along  $u_2$  (vide Fig.23.2c), keeping  $u_1 = 0$ . Calculate the required forces for this configuration  $k_{12}$  and  $k_{22}$ . Invoking the principle of superposition, the forces  $P_1$  and  $P_2$  are expressed as the sum of forces developed due to displacements  $u_1$  and  $u_2$  acting separately on the beam. Thus,

$$P_1 = k_{11}u_1 + k_{12}u_2$$

$$P_2 = k_{21}u_1 + k_{22}u_2 \quad (23.4)$$

$$\{P\} = [k]\{u\}$$

where,  $\{P\} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$ ;  $\{k\} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$ ; and  $\{u\} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$ .

$[k]$  is defined as the stiffness matrix of the beam.

In this lesson, using stiffness method a few problems will be solved. However this approach is very rudimentary and is suited for hand computation. A more formal approach of the stiffness method will be presented in the next lesson.

## 23.2 A simple example with one degree of freedom

Consider a fixed–simply supported beam of constant flexural rigidity  $EI$  and span  $L$  which is carrying a uniformly distributed load of  $w$  kN/m as shown in Fig.23.3a.

If the axial deformation is neglected, then this beam is kinematically indeterminate to first degree. The only unknown joint displacement is  $\theta_B$ . Thus the degrees of freedom for this structure is one (for a brief discussion on degrees of freedom, please see introduction to module 3). The analysis of above structure by stiffness method is accomplished in following steps:

1. Recall that in the flexibility /force method the redundants are released (i.e. made zero) to obtain a statically determinate structure. A similar operation in the stiffness method is to make all the unknown displacements equal to zero by altering the boundary conditions. Such an altered structure is known as kinematically determinate structure as all joint displacements are known in this case. In the present case the restrained structure is obtained by preventing the rotation at  $B$  as shown in Fig.23.3b. Apply all the external loads on the kinematically determinate structure. Due to restraint at  $B$ , a moment  $M_B$  is developed at  $B$ . In the stiffness method we adopt the following sign convention. Counterclockwise moments and counterclockwise rotations are taken as positive, upward forces and displacements are taken as positive. Thus,

$$M_B = -\frac{wl^2}{12} \quad (\text{-ve as } M_B \text{ is clockwise}) \quad (23.5)$$

The fixed end moment may be obtained from the table given at the end of lesson 14.

2. In actual structure there is no moment at  $B$ . Hence apply an equal and opposite moment  $M_B$  at  $B$  as shown in Fig.23.3c. Under the action of ( $-M_B$ ) the joint rotates in the clockwise direction by an unknown amount. It is observed that superposition of above two cases (Fig.23.3b and Fig.23.3c) gives the forces in the actual structure. Thus the rotation of joint



$B$  must be  $\theta_B$  which is unknown. The relation between  $-M_B$  and  $\theta_B$  is established as follows. Apply a unit rotation at  $B$  and calculate the moment. ( $k_{BB}$ ) caused by it. That is given by the relation

$$k_{BB} = \frac{4EI}{L} \quad (23.6)$$

where  $k_{BB}$  is the stiffness coefficient and is defined as the force at joint  $B$  due to unit displacement at joint  $B$ . Now, moment caused by  $\theta_B$  rotation is

$$M_B = k_{BB}\theta_B \quad (23.7)$$

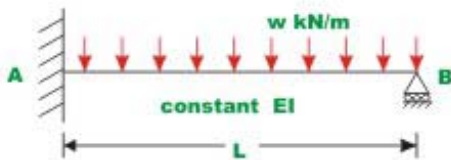
3. Now, write the equilibrium equation for joint  $B$ . The total moment at  $B$  is  $M_B + k_{BB}\theta_B$ , but in the actual structure the moment at  $B$  is zero as support  $B$  is hinged. Hence,

$$M_B + k_{BB}\theta_B = 0 \quad (23.8)$$

$$\theta_B = -\frac{M_B}{k_{BB}}$$

$$\theta_B = \frac{wl^3}{48EI} \quad (23.9)$$

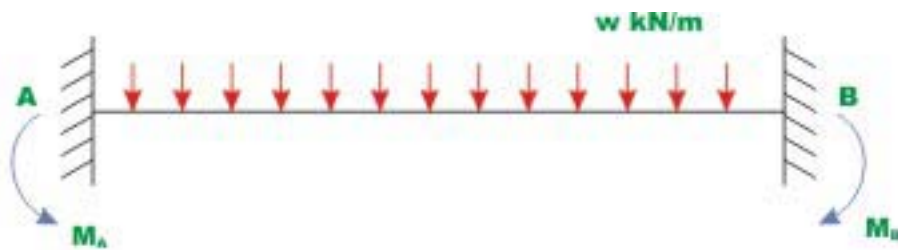
The relation  $M_B = \frac{4EI}{L}\theta_B$  has already been derived in slope-deflection method in lesson 14. Please note that exactly the same steps are followed in slope-deflection method.



**Fig. 23.3(a) Propped - Cantilever beam : one - degree freedom system**



**Fig.23.3(a) Cantilever beam**



**Fig. 23.3b Kinematically determinate beam**



**Fig. 23.3 ©**



**Fig. 23.3 (d) Computation of stiffness co-efficients**

## 23.3 Two degrees of freedom structure

Consider a plane truss as shown in Fig.23.4a. There are four members in the truss and they meet at the common point at  $E$ . The truss is subjected to external loads  $P_1$  and  $P_2$  acting at  $E$ . In the analysis, neglect the self weight of members. There are two unknown displacements at joint  $E$  and are denoted by  $u_1$  and  $u_2$ . Thus the structure is kinematically indeterminate to second degree. The applied forces and unknown joint displacements are shown in the positive directions. The members are numbered from (1), (2), (3) and (4) as shown in the figure. The length and axial rigidity of  $i$ -th member is  $l_i$  and  $EA_i$  respectively. Now it is sought to evaluate  $u_1$  and  $u_2$  by stiffness method. This is done in following steps:

1. In the first step, make all the unknown displacements equal to zero by altering the boundary conditions as shown in Fig.23.4b. On this restrained /kinematically determinate structure, apply all the external loads except the joint loads and calculate the reactions corresponding to unknown joint displacements  $u_1$  and  $u_2$ . Since, in the present case, there are no external loads other than the joint loads, the reactions  $(R_L)_1$  and  $(R_L)_2$  will be equal to zero. Thus,

$$\begin{Bmatrix} (R_L)_1 \\ (R_L)_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (23.10)$$

2. In the next step, calculate stiffness coefficients  $k_{11}, k_{21}, k_{12}$  and  $k_{22}$ . This is done as follows. First give a unit displacement along  $u_1$  holding displacement along  $u_2$  to zero and calculate reactions at  $E$  corresponding to unknown displacements  $u_1$  and  $u_2$  in the kinematically determinate structure. They are denoted by  $k_{11}, k_{21}$ . The joint stiffness  $k_{11}, k_{21}$  of the whole truss is composed of individual member stiffness of the truss. This is shown in Fig.23.4c. Now consider the member  $AE$ . Under the action of unit displacement along  $u_1$ , the joint  $E$  displaces to  $E'$ . Obviously the new length is not equal to length  $AE$ . Let us denote the new length of the members by  $l_1 + \Delta l_1$ , where  $\Delta l_1$  is the change in length of the member  $AE'$ . The member  $AE'$  also makes an angle  $\theta_1$  with the horizontal. This is justified as  $\Delta l_1$  is small. From the geometry, the change in length of the members  $AE'$  is

$$\Delta l_1 = \cos \theta_1 \quad (23.11a)$$

The elongation  $\Delta l_1$  is related to the force in the member  $AE'$ ,  $F_{AE'}$  by

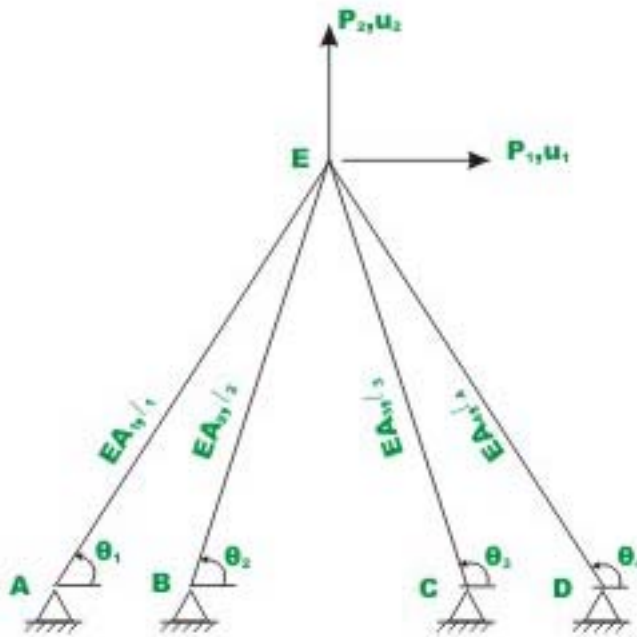
$$\Delta l_1 = \frac{F_{AE'} l_1}{A_1 E} \quad (23.11b)$$

Thus from (23.11a) and (23.11b), the force in the members  $AE'$  is

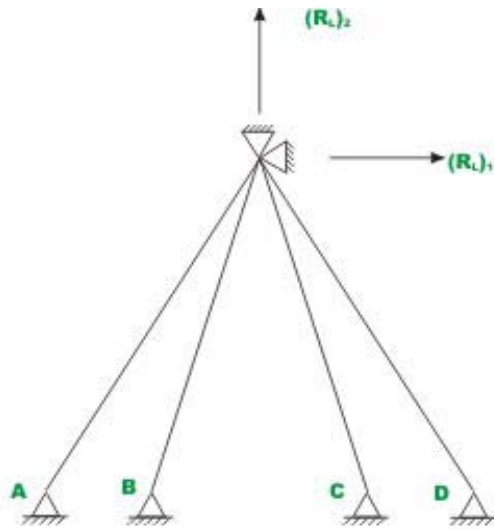
$$F'_{AE} = \frac{EA_1}{l_1} \cos \theta_1 \quad (23.11c)$$

This force acts along the member axis. This force may be resolved along  $u_1$  and  $u_2$  directions. Thus, horizontal component of force  $F'_{AE}$  is  $\frac{EA_1}{l_1} \cos^2 \theta_1$  (23.11d)

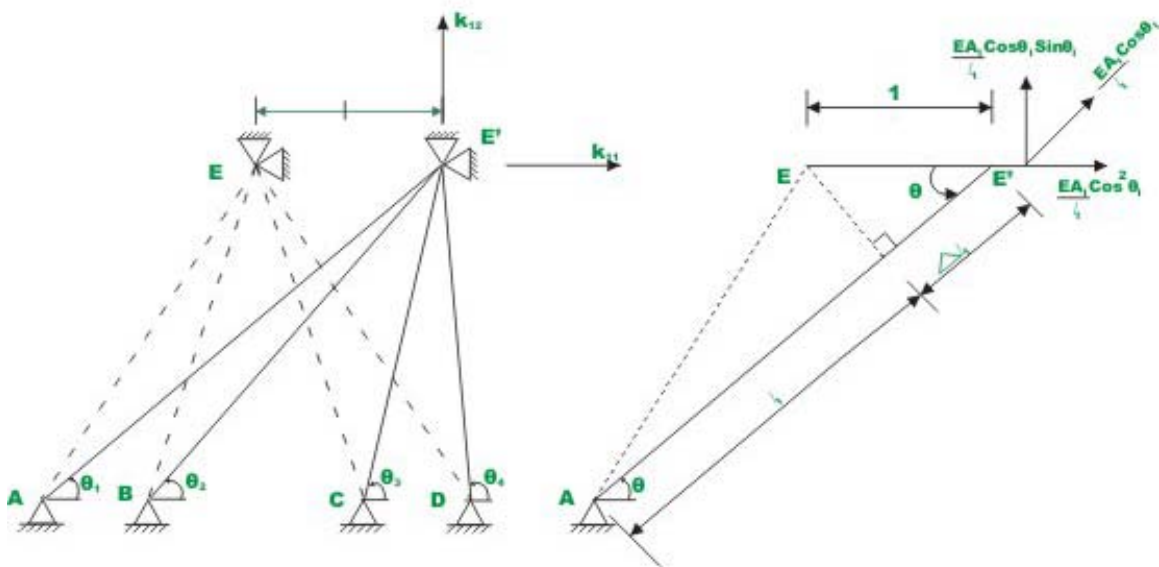
and vertical component of force  $F'_{AE}$  is  $\frac{EA_1}{l_1} \cos \theta_1 \sin \theta_1$  (23.11e)



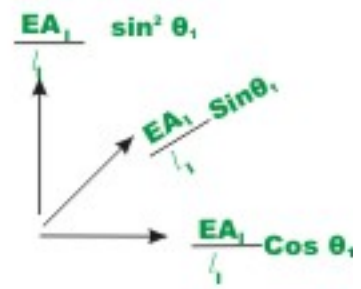
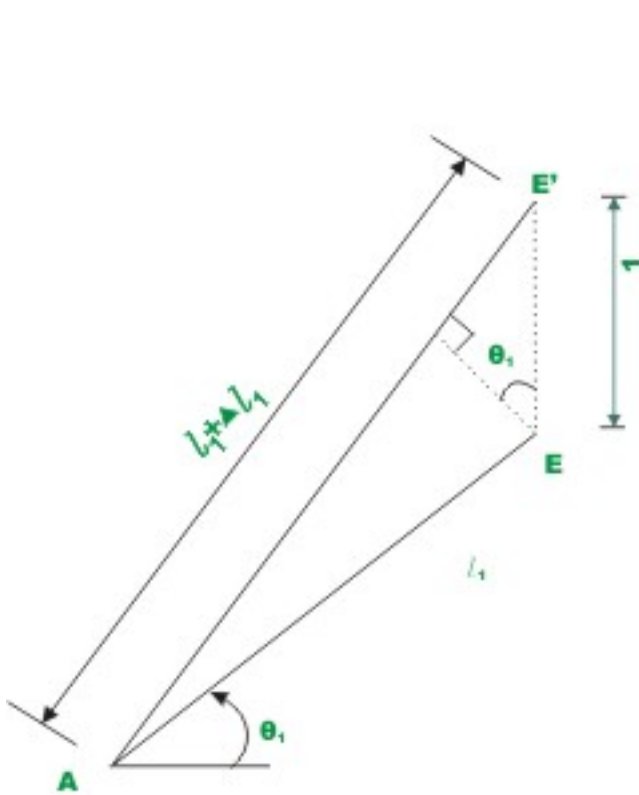
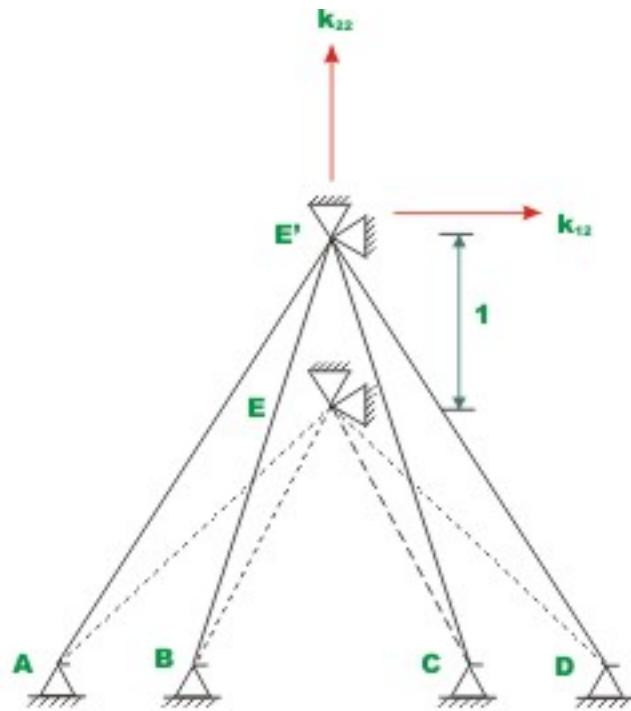
**Fig 23.4a A four - member truss**



**Fig. 23.4b Kinematically determinate structure**



**Fig. 23.4c Unit displacement along  $u_1$**



**Fig.23.4d Unit displacement along  $u_2$**

Expressions of similar form as above may be obtained for all members. The sum of all horizontal components of individual forces gives us the stiffness coefficient  $k_{11}$  and sum of all vertical component of forces give us the required stiffness coefficient  $k_{21}$ .

$$k_{11} = \frac{EA_1}{l_1} \cos^2 \theta_1 + \frac{EA_2}{l_2} \cos^2 \theta_2 + \frac{EA_3}{l_3} \cos^2 \theta_3 + \frac{EA_4}{l_4} \cos^2 \theta_4$$

$$k_{11} = \sum_{i=1}^4 \frac{EA_i}{l_i} \cos^2 \theta_i \quad (23.12)$$

$$k_{21} = \sum \frac{EA_i}{l_i} \cos \theta_i \sin \theta_i \quad (23.13)$$

In the next step, give a unit displacement along  $u_2$  holding displacement along  $u_1$  equal to zero and calculate reactions at  $E$  corresponding to unknown displacements  $u_1$  and  $u_2$  in the kinematically determinate structure. The corresponding reactions are denoted by  $k_{12}$  and  $k_{22}$  as shown in Fig.23.4d. The joint  $E$  gets displaced to  $E'$  when a unit vertical displacement is given to the joint as shown in the figure. Thus, the new length of the member  $AE'$  is  $l_1 + \Delta l_1$ . From the geometry, the elongation  $\Delta l_1$  is given by

$$\Delta l_1 = \sin \theta_1 \quad (23.14a)$$

Thus axial force in the member along its centroidal axis is  $\frac{EA_1}{l_1} \sin \theta_1$  (23.14b)

Resolve the axial force in the member along  $u_1$  and  $u_2$  directions. Thus, horizontal component of force in the member  $AE'$  is  $\frac{EA_1}{l_1} \sin \theta_1 \cos \theta_1$  (23.14c)

and vertical component of force in the member  $AE'$  is  $\frac{EA_1}{l_1} \sin^2 \theta_1$  (23.14d)

In order to evaluate  $k_{22}$ , we need to sum vertical components of forces in all the members meeting at joint  $E$ . Thus,

$$k_{22} = \sum_{i=1}^4 \frac{EA_i}{l_i} \sin^2 \theta_i \quad (23.15)$$

$$\text{Similarly, } k_{12} = \sum_{i=1}^4 \frac{EA_i}{l_i} \sin \theta_i \cos \theta_i \quad (23.16)$$

3. Joint forces in the original structure corresponding to unknown displacements  $u_1$  and  $u_2$  are

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} \quad (23.17)$$

Now the equilibrium equations at joint  $E$  states that the forces in the original structure are equal to the superposition of (i) reactions in the kinematically restrained structure corresponding to unknown joint displacements and (ii) reactions in the restrained structure due to unknown displacements themselves. This may be expressed as,

$$\begin{aligned} F_1 &= (R_L)_1 + k_{11}u_1 + k_{12}u_2 \\ F_2 &= (R_L)_2 + k_{21}u_1 + k_{22}u_2 \end{aligned} \quad (23.18)$$

This may be written compactly as

$$\{F\} = \{R_i\} + [k]\{u\} \quad (23.19)$$

where,

$$\begin{aligned} \{F\} &= \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}; \\ \{R_L\} &= \begin{Bmatrix} (R_L)_1 \\ (R_L)_2 \end{Bmatrix} \\ [k] &= \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \\ \{u\} &= \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \end{aligned} \quad (23.20)$$



For example take  $P_1 = P_2 = P$ ,  $L_i = \frac{L}{\sin \theta_i}$ ,  $A_1 = A_2 = A_3 = A_4 = A$  and  $\theta_1 = 35^\circ$ ,  $\theta_2 = 70^\circ$ ,  $\theta_3 = 105^\circ$  and  $\theta_4 = 140^\circ$

Then.

$$\{F\} = \begin{Bmatrix} P \\ P \end{Bmatrix} \quad (23.21)$$

$$\{R_L\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$k_{11} = \sum \frac{EA}{L} \cos^2 \theta_i \sin \theta_i = 0.9367 \frac{EA}{L}$$

$$k_{12} = \sum \frac{EA}{L} \sin^2 \theta_i \cos \theta_i = 0.0135 \frac{EA}{L}$$

$$k_{21} = \sum \frac{EA}{L} \sin^2 \theta_i \cos \theta_i = 0.0135 \frac{EA}{L}$$

$$k_{22} = \sum \frac{EA}{L} \sin^3 \theta_i = 2.1853 \frac{EA}{L} \quad (23.22)$$

$$\begin{Bmatrix} P \\ P \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 0.9367 & 0.0135 \\ 0.0135 & 2.1853 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

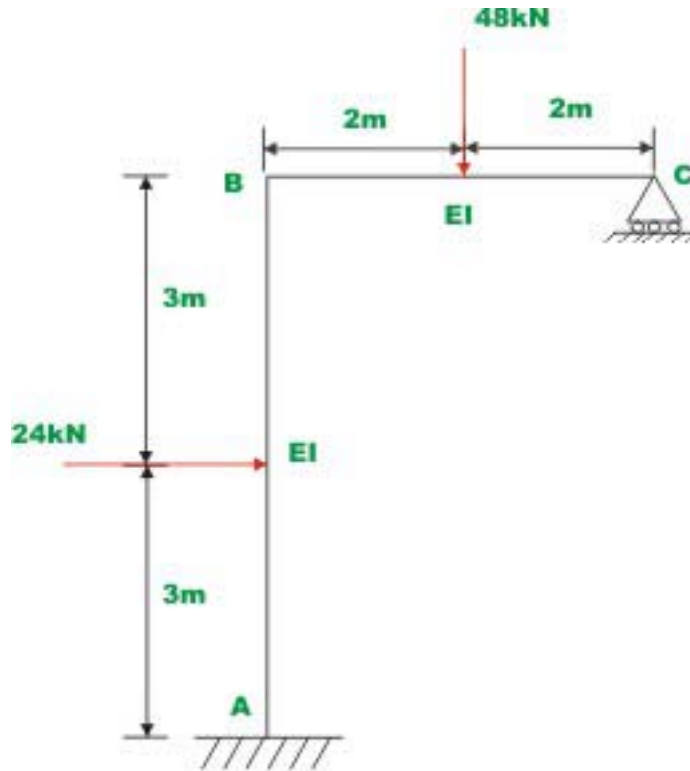
Solving which, yields

$$u_1 = 1.0611 \frac{L}{EA}$$

$$u_2 = 0.451 \frac{L}{EA}$$

### Example 23.1

Analyze the plane frame shown in Fig.23.5a by the direct stiffness method. Assume that the flexural rigidity for all members is the same. Neglect axial displacements.



**Fig 23.5a Plane - frame of Example 23.1**

### Solution

In the first step identify the degrees of freedom of the frame. The given frame has three degrees of freedom (see Fig.23.5b):

- (i) Two rotations as indicated by  $u_1$  and  $u_2$  and
- (ii) One horizontal displacement of joint B and C as indicated by  $u_3$ .

In the next step make all the displacements equal to zero by fixing joints B and C as shown in Fig.23.5c. On this kinematically determinate structure apply all the external loads and calculate reactions corresponding to unknown joint displacements. Thus,

$$\left(R_D^F\right)_1 = \frac{48 \times 2 \times 4}{16} + \left(-\frac{24 \times 3 \times 9}{36}\right)$$

(1)

$$= 24 - 18 = 6 \text{ kN.m}$$

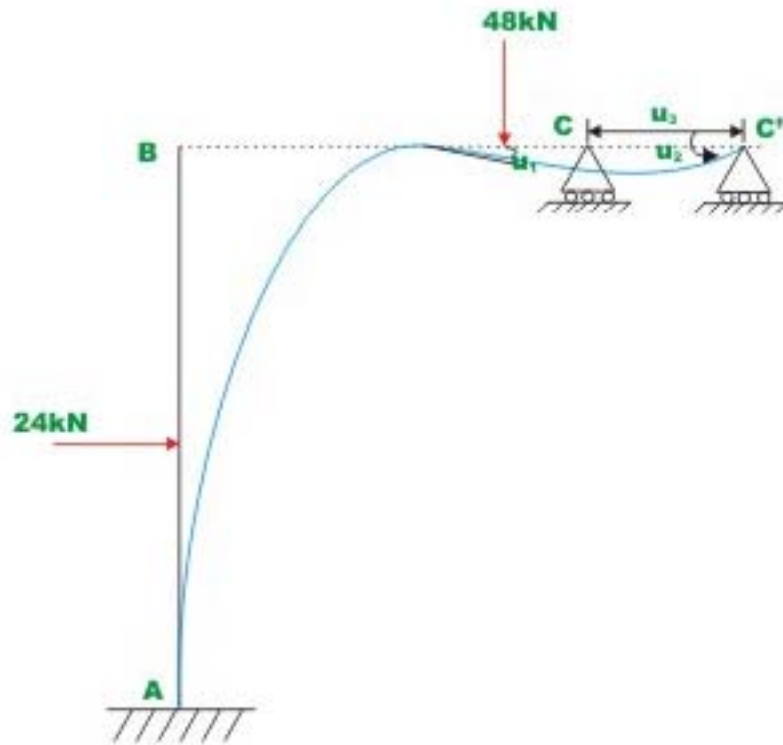
$$\left(R_D^F\right)_2 = -24 \text{ kN.m}$$

$$\left(R_D^F\right)_3 = 12 \text{ kN.m} \quad (2)$$

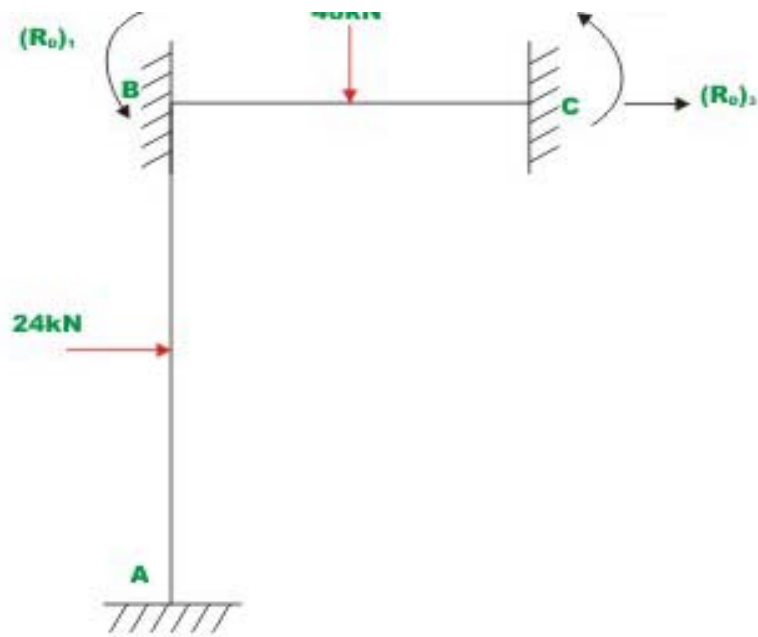
Thus,

$$\begin{Bmatrix} \left(R_D^F\right)_1 \\ \left(R_D^F\right)_2 \\ \left(R_D^F\right)_3 \end{Bmatrix} = \begin{Bmatrix} 6 \\ -24 \\ 12 \end{Bmatrix} \quad (3)$$

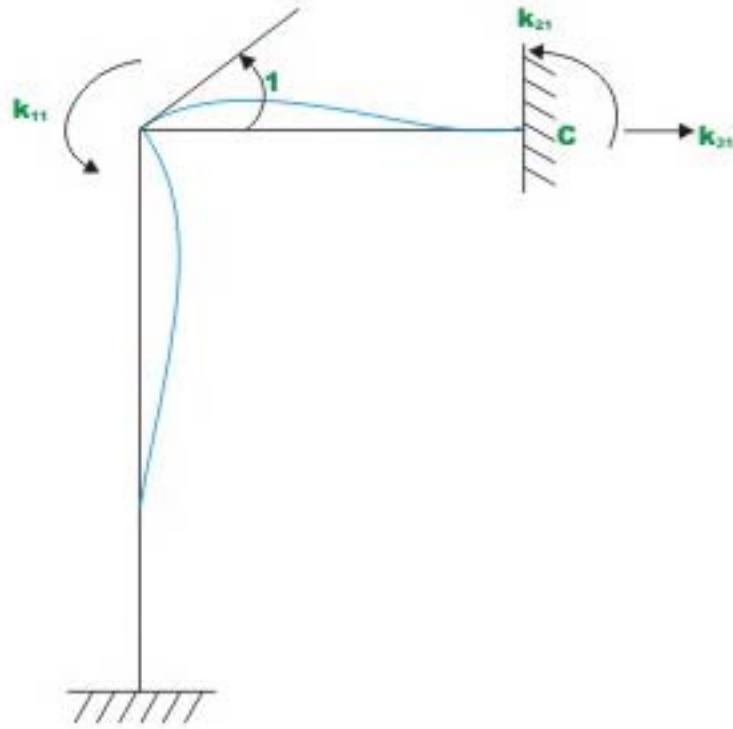
Next calculate stiffness coefficients. Apply unit rotation along  $u_1$  and calculate reactions corresponding to the unknown joint displacements in the kinematically determinate structure (vide Fig.23.5d)



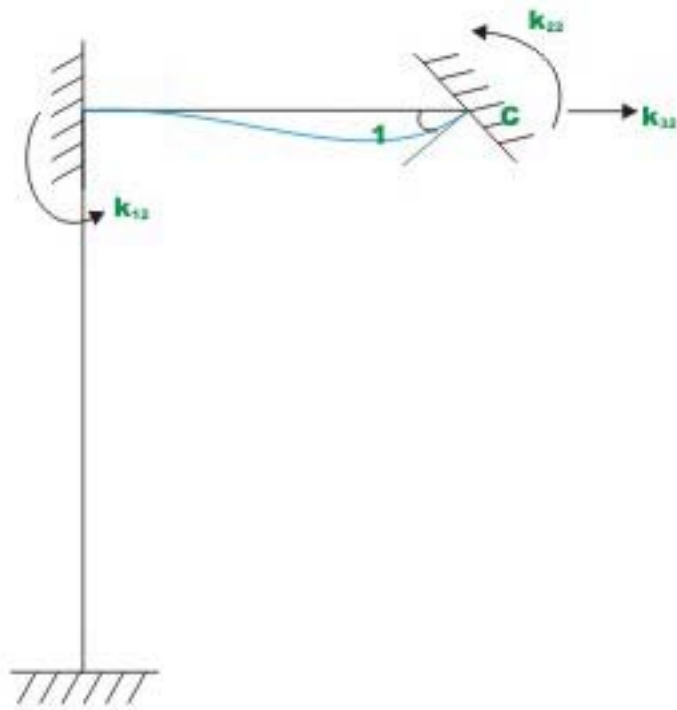
**Fig 23.5b Approximate deflected shape**



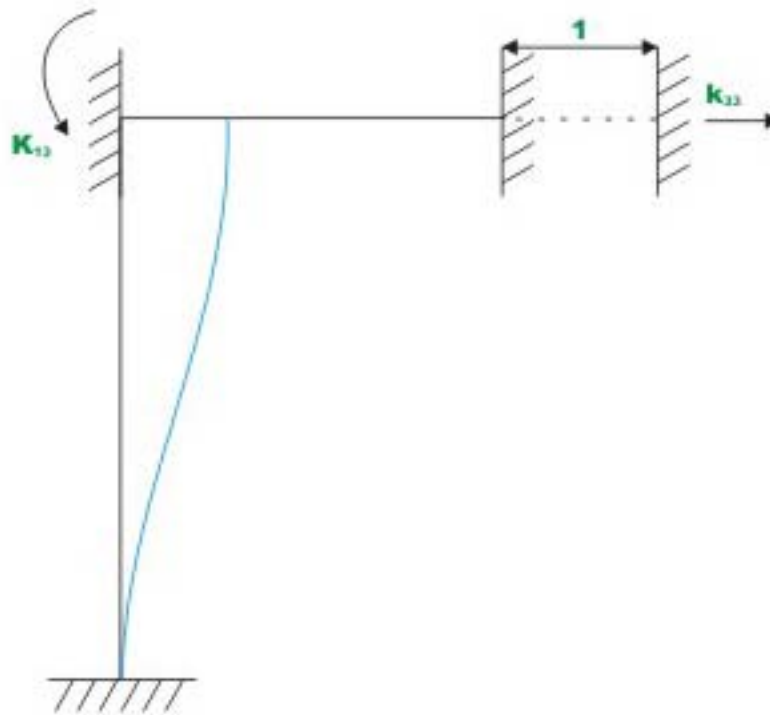
**Fig 23.5c Kinematically restrained structure**



**Fig.23.5d Unit displacement along  $u_1$**



**Fig 23.5e Unit displacement along  $u_2$**



**Fig. 23.5f Unit displacement along  $u_3$**

$$k_{11} = \frac{4EI}{4} + \frac{4EI}{6} = 1.667$$

$$k_{21} = \frac{2EI}{4} = 0.5EI$$

$$k_{31} = -\frac{6EI}{6 \times 6} = -0.166EI \quad (4)$$

Similarly, apply a unit rotation along  $u_2$  and calculate reactions corresponding to three degrees of freedom (see Fig.23.5e)

$$k_{12} = 0.5EI$$

$$k_{22} = EI$$

$$k_{32} = 0 \quad (5)$$

Apply a unit displacement along  $u_3$  and calculate joint reactions corresponding to unknown displacements in the kinematically determinate structure.

$$k_{13} = -\frac{6EI}{L^2} = -0.166E$$

$$k_{23} = 0$$

$$k_{33} = \frac{12EI}{6^3} = 0.056EI \quad (6)$$

Finally applying the principle of superposition of joint forces, yields

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 6 \\ -24 \\ 12 \end{Bmatrix} + EI \begin{Bmatrix} 1.667 & 0.5 & -0.166 \\ 0.5 & 1 & 0 \\ -0.166 & 0 & 0.056 \end{Bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

Now,

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \text{ as there are no loads applied along } u_1, u_2 \text{ and } u_3. \text{ Thus the}$$

unknown displacements are,

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = -\frac{1}{EI} \begin{bmatrix} 1 & 0.5 & -0.166 \\ 0.5 & 1 & 0 \\ -0.166 & 0 & 0.056 \end{bmatrix}^{-1} \begin{Bmatrix} 6 \\ -24 \\ -24 \end{Bmatrix} \quad (7)$$

Solving

$$u_1 = \frac{18.996}{EI}$$

$$u_2 = \frac{14.502}{EI}$$

$$u_3 = -\frac{270.587}{EI} \quad (8)$$

## Summary

The flexibility coefficient and stiffness coefficients are defined in this section. Construction of stiffness matrix for a simple member is explained. A few simple problems are solved by the direct stiffness method. The difference between the slope-deflection method and the direct stiffness method is clearly brought out.