Module
3
Analysis of Statically Indeterminate Structures by the Displacement Method

Lesson 21

The Moment-Distribution Method: Frames with Sidesway

Instructional Objectives

After reading this chapter the student will be able to

1. Extend moment-distribution method for frames undergoing sidesway.
2. Draw free-body diagrams of plane frame.
3. Analyse plane frames undergoing sidesway by the moment-distribution method.
4. Draw shear force and bending moment diagrams.
5. Sketch deflected shape of the plane frame not restrained against sidesway.

21.1 Introduction

In the previous lesson, rigid frames restrained against sidesway are analyzed using moment-distribution method. It has been pointed in lesson 17, that frames which are unsymmetrical or frames which are loaded unsymmetrically usually get displaced either to the right or to the left. In other words, in such frames apart from evaluating joint rotations, one also needs to evaluate joint translations (sidesway). For example in frame shown in Fig 21.1, the loading is symmetrical but the geometry of frame is unsymmetrical and hence sidesway needs to be considered in the analysis. The number of unknowns is this case are: joint rotations θ_B and θ_C and member rotation ψ. Joint B and C get translated by the same amount as axial deformations are not considered and hence only one independent member rotation need to be considered. The procedure to analyze rigid frames undergoing lateral displacement using moment-distribution method is explained in section 21.2 using an example.

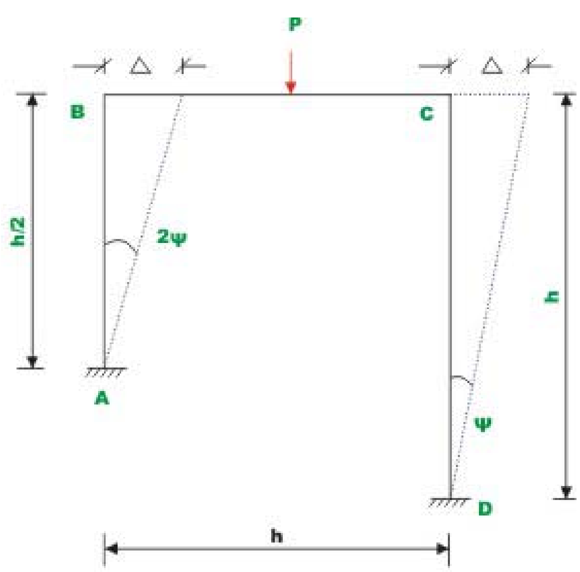

Fig 21.1 Rigid frame

21.2 Procedure

A special procedure is required to analyze frames with sidesway using moment-distribution method. In the first step, identify the number of independent rotations (ψ) in the structure. The procedure to calculate independent rotations is explained in lesson 22. For analyzing frames with sidesway, the method of superposition is used. The structure shown in Fig. 21.2a is expressed as the sum of two systems: Fig. 21.2b and Fig. 21.2c. The systems shown in figures 21.2b and 21.2c are analyzed separately and superposed to obtain the final answer. In system 21.2b, sidesway is prevented by artificial support at C. Apply all the external loads on frame shown in Fig. 21.2b. Since for the frame, sidesway is prevented, moment-distribution method as discussed in the previous lesson is applied and beam end moments are calculated. Let $M'_{AB}, M'_{BA}, M'_{BC}, M'_{CB}, M'_{CD}$ and M'_{DC} be the balanced moments obtained by distributing fixed end moments due to applied loads while allowing only joint rotations (θ_B and θ_C) and preventing sidesway.

Now, calculate reactions H_{A1} and H_{D1} (ref. Fig 21.3a).they are ,

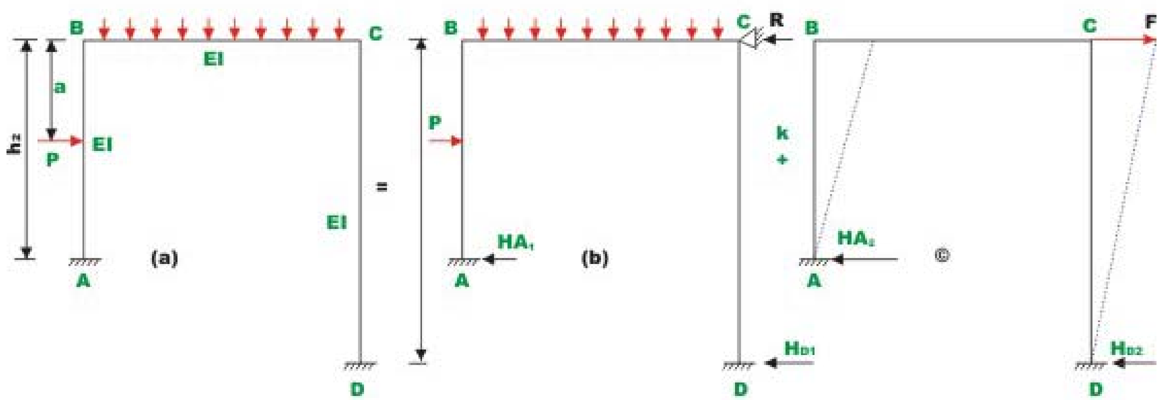

Fig 21.2 Frame with sidesway

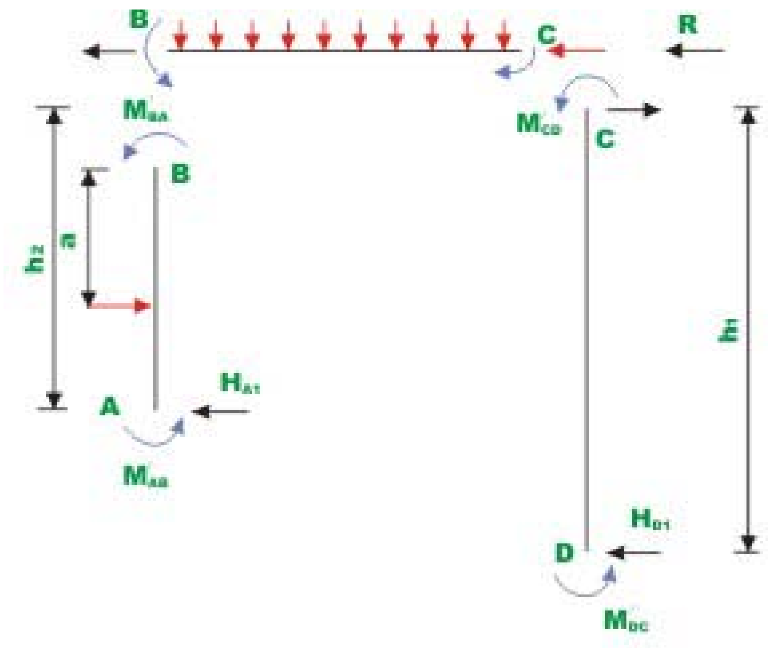

Fig.21.3a Free body diagram

$$H_{A1} = \frac{M'_{AB} + M'_{BA}}{h_2} + \frac{Pa}{h_2}$$

$$H_{D1} = \frac{M'_{CD} + M'_{DC}}{h_1} \qquad (21.1)$$

again, $$R = P - (H_{A1} + H_{D1}) \qquad (21.2)$$

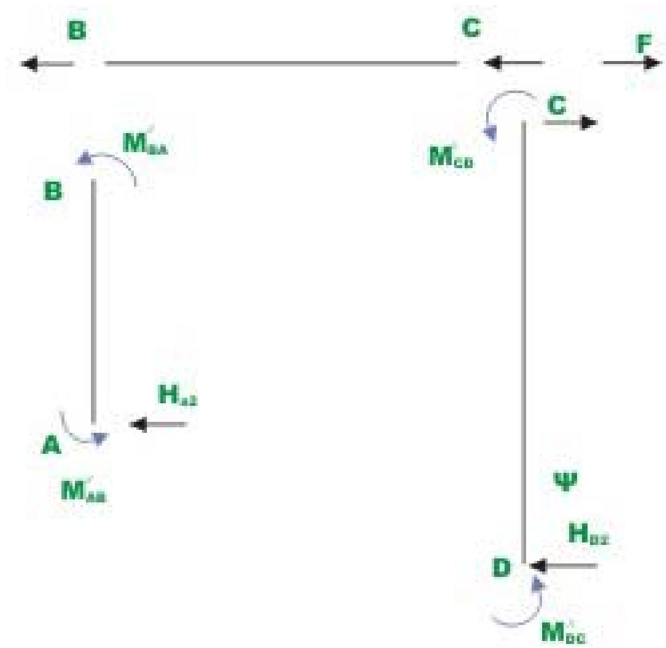

Fig.21.3b Free body diagram of frame

In Fig 21.2c apply a horizontal force F in the opposite direction of R. Now $kF = R$, then the superposition of beam end moments of system (b) and k times (c) gives the results for the original structure. However, there is no way one could analyze the frame for horizontal force F, by moment-distribution method as sway comes in to picture. Instead of applying F, apply arbitrary known displacement / sidesway Δ' as shown in the figure. Calculate the fixed end beam moments in the column AB and CD for the imposed horizontal displacement. Since joint displacement is known beforehand, one could use moment-distribution method to analyse this frame. In this case, member rotations ψ are related to joint translation which is known. Let $M''_{AB}, M''_{BA}, M''_{BC}, M''_{CB}, M''_{CD}$ and M''_{DC} are the balanced moment obtained by distributing the fixed end moments due to assumed sidesway Δ' at joints B and C. Now, from statics calculate horizontal force F due to arbitrary sidesway Δ'.

$$H_{A2} = \frac{M''_{AB} + M''_{BA}}{h_2}$$

$$H_{D2} = \frac{M''_{CD} + M''_{DC}}{h_1} \qquad (21.3)$$

$$F = (H_{A2} + H_{D2}) \qquad (21.4)$$

In Fig 21.2, by method of superposition

$kF = R$ or $k = R / F$

Substituting the values of R and F from equations (21.2) and (21.4),

$$k = \frac{P - (H_{A1} + H_{D1})}{(H_{A2} + H_{D2})} \qquad (21.5)$$

Now substituting the values of H_{A1}, H_{A2}, H_{D1} and H_{D2} in 21.5,

$$k = \frac{P - \left(\frac{M'_{AB} + M'_{BA}}{h_2} + \frac{Pa}{h_2} \right) + \frac{M'_{CD} + M'_{DC}}{h_1}}{\left(\frac{M''_{AB+} M''_{BA}}{h_2} + \frac{M''_{CD} + M''_{DC}}{h_1} \right)} \qquad (21.6)$$

Hence, beam end moment in the original structure is obtained as,

$$M_{original} = M_{system(b)} + kM_{system(c)}$$

If there is more than one independent member rotation, then the above procedure needs to be modified and is discussed in the next lesson.

Example 21.1

Analyse the rigid frame shown in Fig 21.4a. Assume EI to be constant for all members. Also sketch elastic curve.

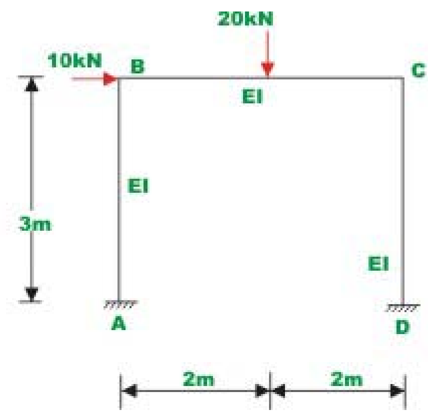

Fig. 21.4a Rigid frame of Example 21.1

Solution

In the given problem, joint C can also rotate and also translate by an unknown amount Δ. This problem has to be solved in two steps. In the first step, evaluate the beam-end moment by preventing the sidesway.

In the second step calculate beam end moments by moment-distribution method for known translation (see Fig 21.4b). By appropriately superposing the two results, the beam end moment of the original structure is obtained.

a) Calculate stiffness and distribution factors

$$K_{BA} = 0.333EI \;;\; K_{BC} = 0.25EI \;;$$

$$K_{CB} = 0.25EI \;;\; K_{CD} = 0.333EI$$

Joint B: $\sum K = 0.583EI$

$$DF_{BA} = 0.571;\; DF_{BC} = 0.429$$

Joint C: $\sum K = 0.583EI$

$$DF_{CB} = 0.429\,;\;\; DF_{CD} = 0.571\,. \qquad (1)$$

b) Calculate fixed end moment due to applied loading.

$$M^F_{AB} = 0\,; \qquad M^F_{BA} = 0 \text{ kN.m}$$

$$M^F_{BC} = +10 \text{ kN.m}\,; \quad M^F_{CB} = -10 \text{ kN.m}$$

$$M^F_{CD} = 0 \text{ kN.m} \quad ; \quad M^F_{DC} = 0 \text{ kN.m}\,. \qquad (2)$$

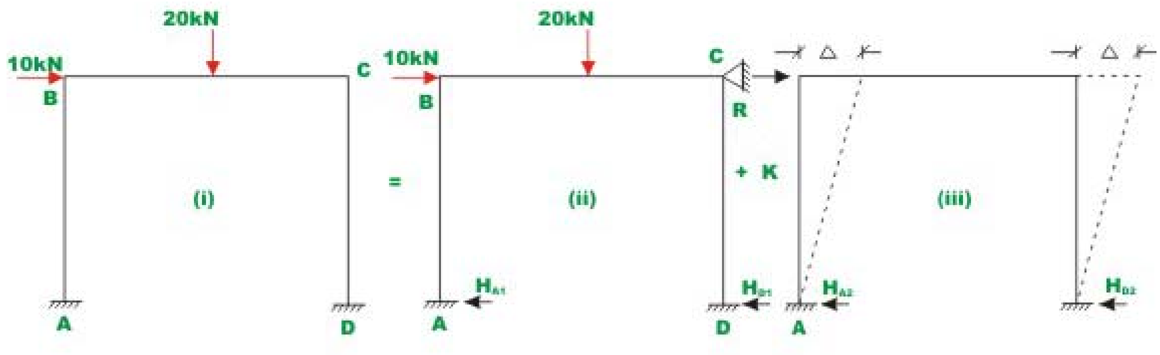

Fig. 21.4b Frame with side - sway

Now the frame is prevented from sidesway by providing a support at C as shown in Fig 21.4b (ii). The moment-distribution for this frame is shown in Fig 21.4c. Let $M'_{AB}, M'_{BA}, M'_{CD}$ and M'_{DC} be the balanced end moments. Now calculate horizontal reactions at A and D from equations of statics.

$$H_{A1} = \frac{M'_{AB} + M'_{BA}}{3}$$

$$= \frac{-3.635 + 7.268}{3}$$

$$= -3.635 \; KN(\rightarrow)\,.$$

$$H_{D1} = \frac{3.636 - 17.269}{3} = 3.635 \text{ kN}(\leftarrow)\,.$$

$$R = 10 - (-3.635 + 3.635) = -10 \ \text{kN}(\rightarrow) \quad (3)$$

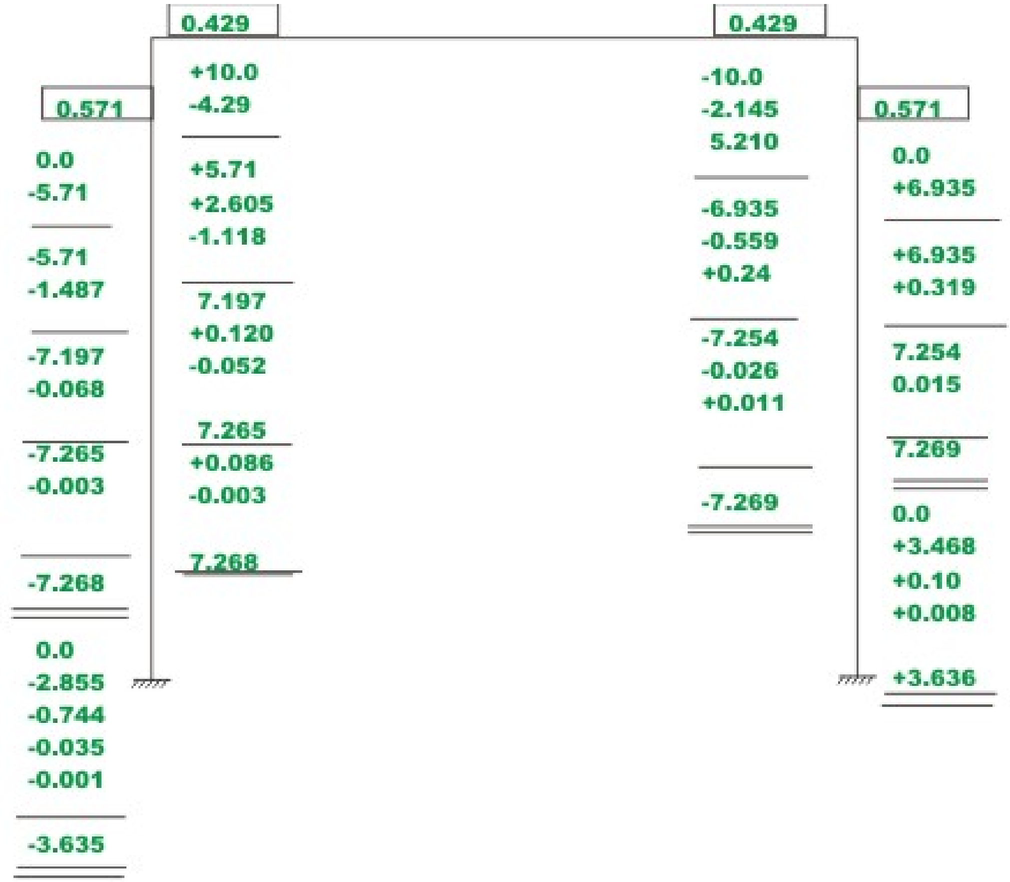

Fig. 21.4c Moment distribution with sidesway prevented

d) Moment-distribution for arbitrary known sidesway Δ' .

Since Δ' is arbitrary, Choose any convenient value. Let $\Delta' = \dfrac{150}{EI}$ Now calculate fixed end beam moments for this arbitrary sidesway.

$$M_{AB}^{F} = -\frac{6EI\psi}{L} = -\frac{6EI}{3} \times (-\frac{150}{3EI}) = 100 \ \text{kN.m}$$

$$M_{BA}^{F} = 100 \ \text{kN.m}$$

$$M_{CD}^{F} = M_{DC}^{F} = +100 \, \text{kN.m} \quad (4)$$

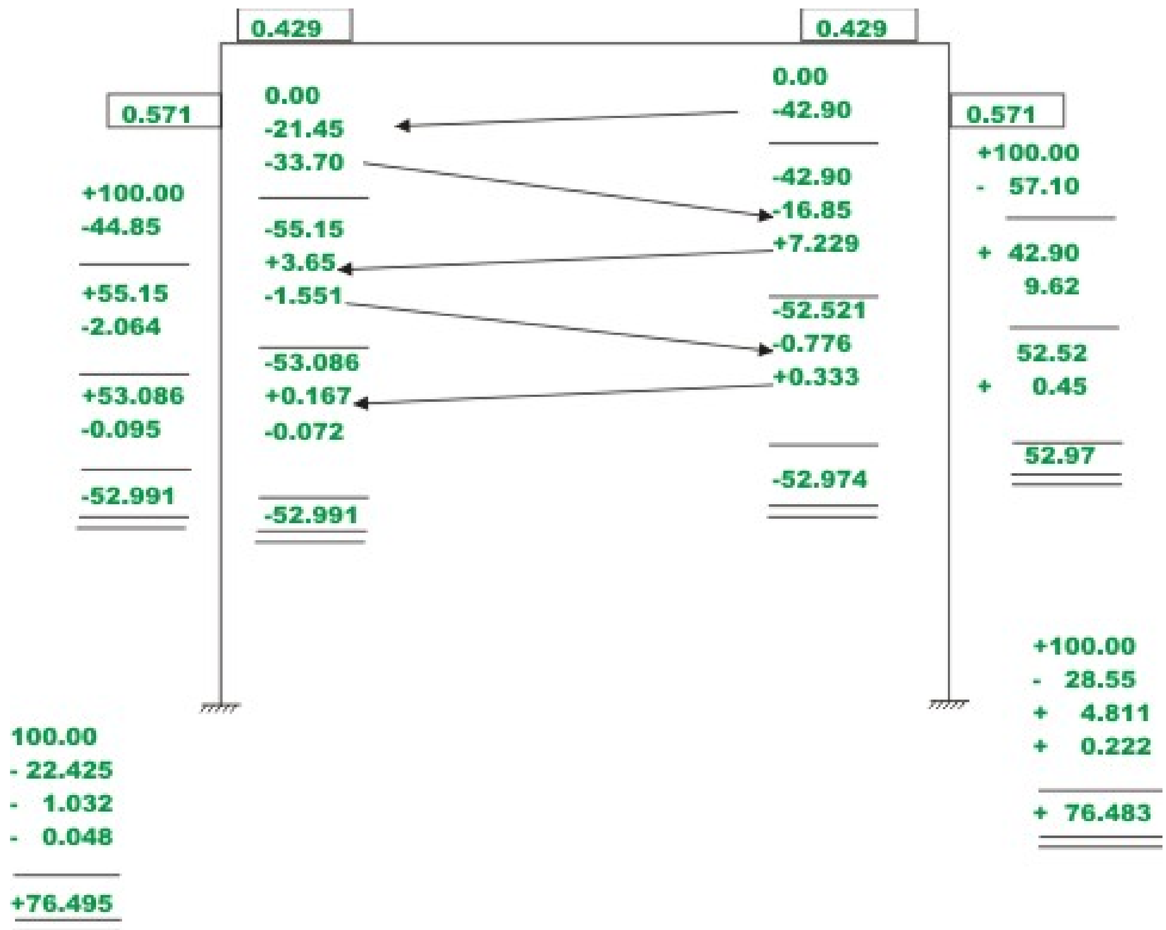

Fig. 21.4d Moment distribution for sidesway

The moment-distribution for this case is shown in Fig 24.4d. Now calculate horizontal reactions H_{A2} and H_{D2}.

$$H_{A2} = \frac{52.98 + 76.48}{3} = 43.15 \text{ kN}(\leftarrow)$$

$$H_{D2} = \frac{52.97 + 76.49}{3} = 43.15 \text{kN}(\leftarrow)$$

$$F = -86.30 \text{ kN}(\rightarrow)$$

Let k be a factor by which the solution of case (iii) needs to be multiplied. Now actual moments in the frame is obtained by superposing the solution (ii) on the solution obtained by multiplying case (iii) by k. Thus kF cancel out the holding force R such that final result is for the frame without holding force.

Thus, $k\,F = R$.

$$k = \frac{-10}{-86.13} = 0.1161 \qquad (5)$$

Now the actual end moments in the frame are,

$$M_{AB} = M'_{AB} + k\,M''_{AB}$$

$$M_{AB} = -3.635 + 0.1161(+76.48) = +5.244 \quad \text{kN.m}$$

$$M_{BA} = -7.268 + 0.1161(+52.98) = -1.117 \quad \text{kN.m}$$

$$M_{BC} = +7.268 + 0.1161(-52.98) = +1.117 \quad \text{kN.m}$$

$$M_{CB} = -7.269 + 0.1161(-52.97) = -13.419 \quad \text{kN.m}$$

$$M_{CD} = +7.268 + 0.1161(+52.97) = +13.418 \quad \text{kN.m}$$

$$M_{DC} = +3.636 + 0.1161(+76.49) = +12.517 \quad \text{kN.m}$$

The actual sway is computed as,

$$\Delta = k\Delta' = 0.1161 \times \frac{150}{EI}$$

$$= \frac{17.415}{EI}$$

The joint rotations can be calculated using slope-deflection equations.

$$M_{AB} = M_{AB}^{F} + \frac{2EI}{L}\left[2\theta_A + \theta_B - 3\psi_{AB}\right] \qquad \text{where } \psi_{AB} = -\frac{\Delta}{L}$$

$$M_{BA} = M_{BA}^{F} + \frac{2EI}{L}\left[2\theta_B + \theta_A - 3\psi_{AB}\right]$$

In the above equation, except θ_A and θ_B all other quantities are known. Solving for θ_A and θ_B,

$$\theta_A = 0 \,; \qquad \theta_B = \frac{-9.55}{EI}.$$

The elastic curve is shown in Fig. 21.4e.

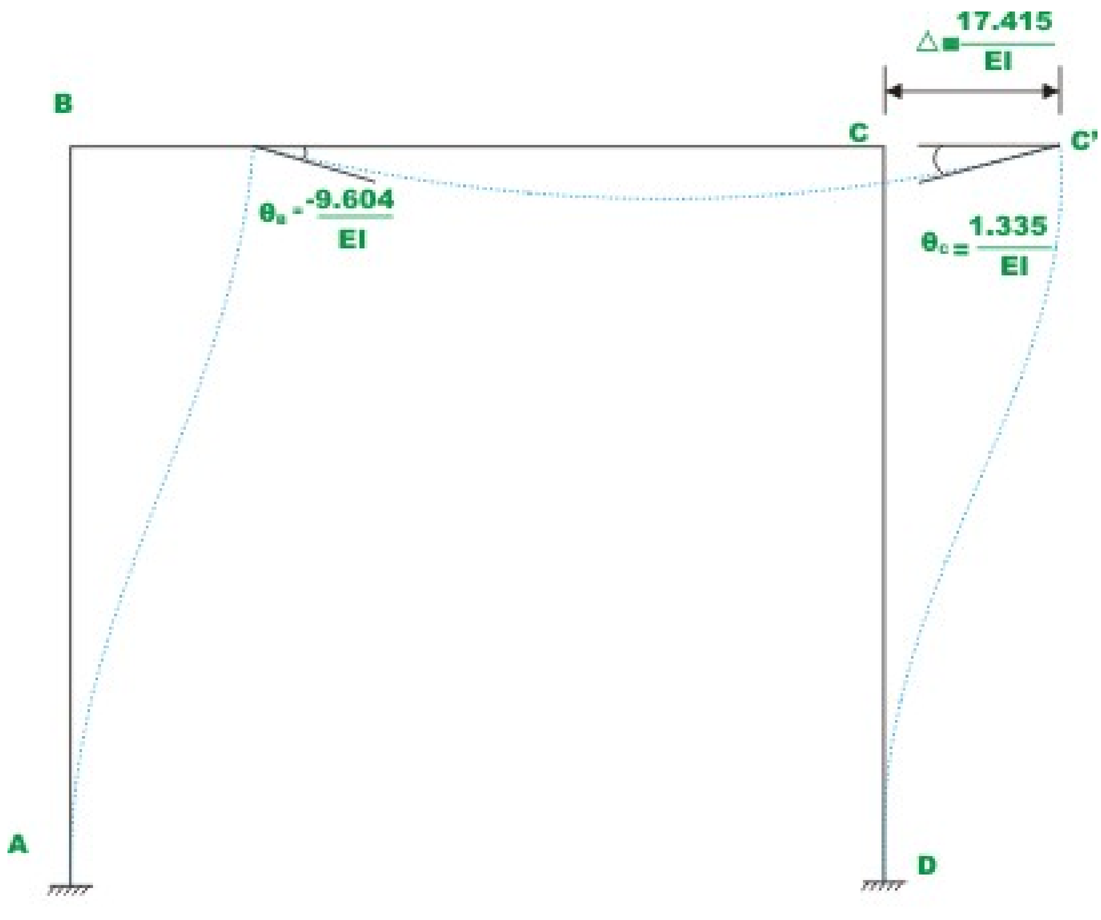

Fig.21.4e Elastic curve

Example 21.2

Analyse the rigid frame shown in Fig. 21.5a by moment-distribution method. The moment of inertia of all the members is shown in the figure. Neglect axial deformations.

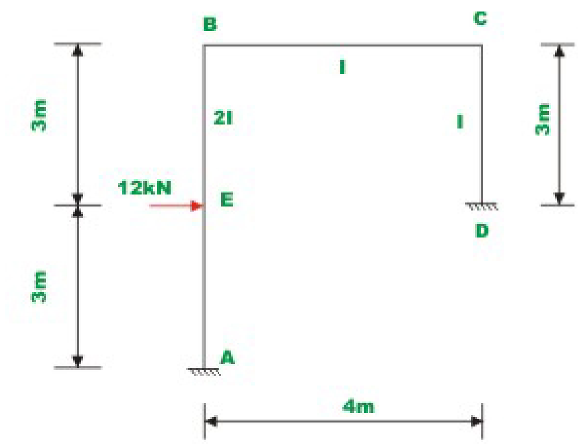

Fig. 21.5a Example 21.2

Solution:

In this frame joint rotations B and C and translation of joint B and C need to be evaluated.

a) Calculate stiffness and distribution factors.

$$K_{BA} = 0.333EI \; ; \qquad K_{BC} = 0.25EI$$

$$K_{CB} = 0.25EI \; ; \qquad K_{CD} = 0.333EI$$

At joint B:

$$\sum K = 0.583EI$$

$$DF_{BA} = 0.571 \; ; \qquad DF_{BC} = 0.429$$

At joint C:

$$\sum K = 0.583EI$$

$$DF_{CB} = 0.429 \; ; \qquad DF_{CD} = 0.571$$

b) Calculate fixed end moments due to applied loading.

$$M_{AB}^{F} = \frac{12 \times 3 \times 3^2}{6^2} = 9.0 \quad \text{kN.m} \; ; M_{BA}^{F} = -9.0 \quad \text{kN.m}$$

$$M_{BC}^{F} = 0 \quad \text{kN.m} \; ; \qquad M_{CB}^{F} = 0 \quad \text{kN.m}$$

$$M_{CD}^{F} = 0 \quad \text{kN.m} \; ; \qquad M_{DC}^{F} = 0 \quad \text{kN.m}$$

c) Prevent sidesway by providing artificial support at C. Carry out moment-distribution (*i.e.* Case A in Fig. 21.5b). The moment-distribution for this case is shown in Fig. 21.5c.

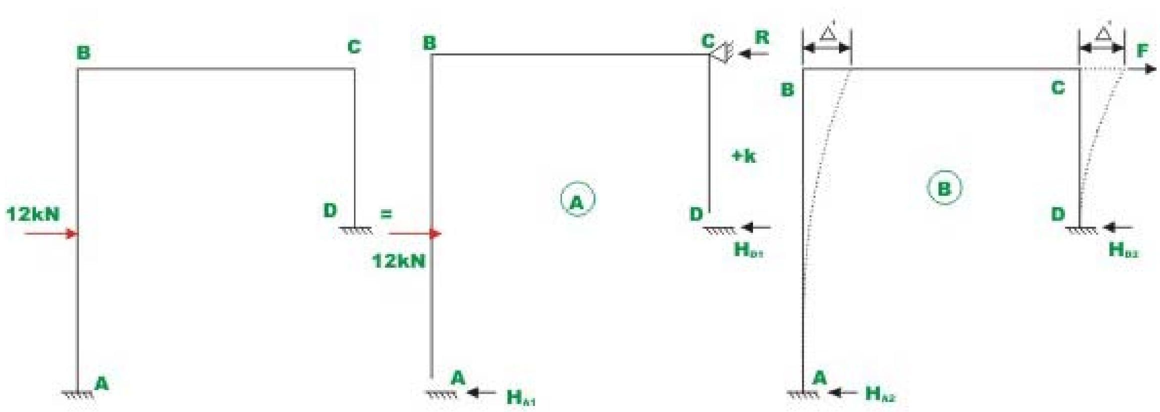

Fig. 21.5 b Frame with sidesway

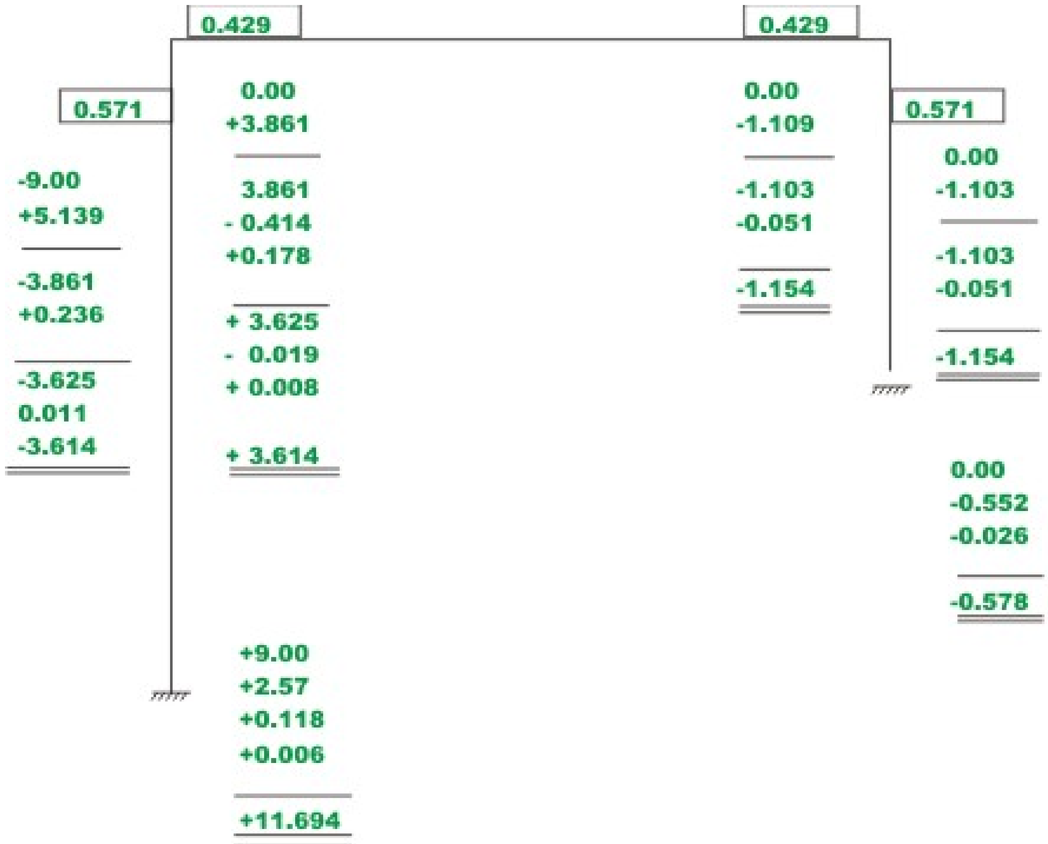

Fig. 21.5c Moment distribution with sidesway prevented

Now calculate horizontal reaction at A and D from equations of statics.

$$H_{A1} = \frac{11.694 - 3.614}{6} + 6 = 7.347 \quad \text{kN}(\leftarrow)$$

$$H_{D1} = \frac{-1.154 - 0.578}{3} = -0.577 \quad \text{kN}(\rightarrow)$$

$$R = 12 - (7.347 - 0.577) = -5.23 \quad \text{kN}(\rightarrow)$$

d) Moment-distribution for arbitrary sidesway Δ' (case B, Fig. 21.5c)

Calculate fixed end moments for the arbitrary sidesway of $\Delta' = \frac{150}{EI}$.

$$M_{AB}^{F} = -\frac{6E(2I)}{L}\psi = \frac{12EI}{6} \times (-\frac{150}{6EI}) = +50 \quad \text{kN.m} \quad ; \qquad M_{BA}^{F} = +50 \quad \text{kN.m} \quad ;$$

$$M^F_{CD} = -\frac{6E(I)}{L}\psi = -\frac{6EI}{3}\times(-\frac{150}{3EI}) = +100 \quad \text{kN.m} \quad ; \qquad M^F_{DC} = +100 \quad \text{kN.m} \quad ;$$

The moment-distribution for this case is shown in Fig. 21.5d. Using equations of static equilibrium, calculate reactions H_{A2} and H_{D2}.

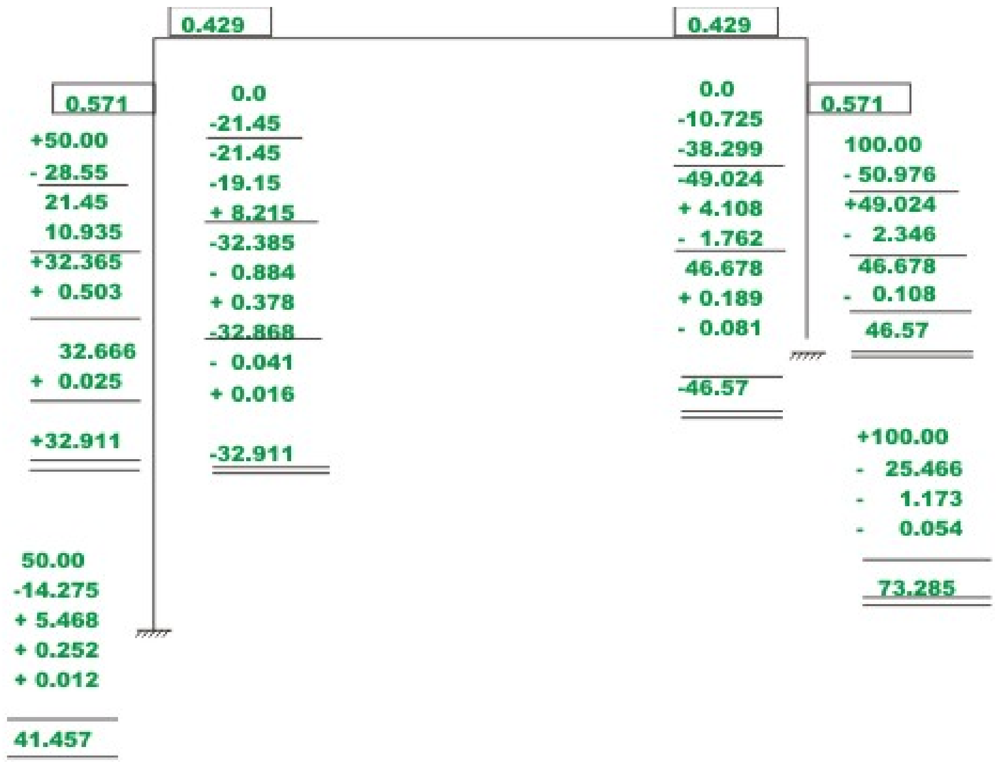

Fig. 21.5d Moment Distribution for arbitrary known sidesway

$$H_{A2} = \frac{32.911 + 41.457}{6} = 12.395 \ kN(\leftarrow)$$

$$H_{D2} = \frac{46.57 + 73.285}{3} = 39.952 \, kN(\leftarrow)$$

$$F = -(12.395 + 39.952) = -52.347 \ kN(\rightarrow)$$

e) Final results

Now, the shear condition for the frame is (vide Fig. 21.5b)

$$(H_{A1} + H_{D1}) + k(H_{A2} + H_{D2}) = 12$$

$$(7.344 - 0.577) + k(12.395 + 39.952) = 12$$

$$k = 0.129$$

Now the actual end moments in the frame are,

$$M_{AB} = M'_{AB} + k\,M''_{AB}$$

$$M_{AB} = 11.694 + 0.129(+41.457) = +17.039 \quad \text{kN.m}$$

$$M_{BA} = -3.614 + 0.129(+32.911) = 0.629 \quad \text{kN.m}$$

$$M_{BC} = 3.614 + 0.129(-32.911) = -0.629 \quad \text{kN.m}$$

$$M_{CB} = -1.154 + 0.129(-46.457) = -4.853 \quad \text{kN.m}$$

$$M_{CD} = -1.154 + 0.129(+46.457) = +4.853 \quad \text{kN.m}$$

$$M_{DC} = -0.578 + 0.129(+73.285) = +8.876 \quad \text{kN.m}$$

The actual sway

$$\Delta = k\,\Delta' = 0.129 \times \frac{150}{EI}$$

$$= \frac{19.35}{EI}$$

The joint rotations can be calculated using slope-deflection equations.

$$M_{AB} - M_{AB}^{F} = +\frac{2E(2I)}{L}\left[2\theta_A + \theta_B - 3\psi\ \right]$$

or

$$\left[2\theta_A + \theta_B\right] = \frac{L}{4EI}\left[M_{AB} - M_{AB}^{F} + \frac{12EI\psi}{L}\right] = \frac{L}{4EI}\left[M_{AB} - \left(M_{AB}^{F} - \frac{12EI\psi}{L}\right)\right]$$

$$\left[2\theta_B + \theta_A\right] = \frac{L}{4EI}\left[M_{BA} - M_{BA}^{F} + \frac{12EI\psi}{L}\right] = \frac{L}{4EI}\left[M_{BA} - \left(M_{BA}^{F} - \frac{12EI\psi}{L}\right)\right]$$

$$M_{AB} = +17.039 \quad \text{kN.m}$$

$$M_{BA} = 0.629 \quad \text{kN.m}$$

$$\left(M_{AB}^{F}\right) = 9 + 0.129(50) = 15.45 \quad \text{kN.m}$$

$$\left(M_{BA}^{F}\right) = -9 + 0.129(50) = -2.55 \quad \text{kN.m}$$

$$\theta_A = \frac{\text{change in near end} + \left(-\frac{1}{2}\right)\text{change in far end}}{3EI/L}$$

$$= \frac{(17.039 - 15.45) + \left(-\frac{1}{2}\right)(0.629 + 2.55)}{3EI/6} = 0.0$$

$$\theta_B = \frac{4.769}{EI}$$

Example 21.3

Analyse the rigid frame shown in Fig. 21.6a. The moment of inertia of all the members are shown in the figure.

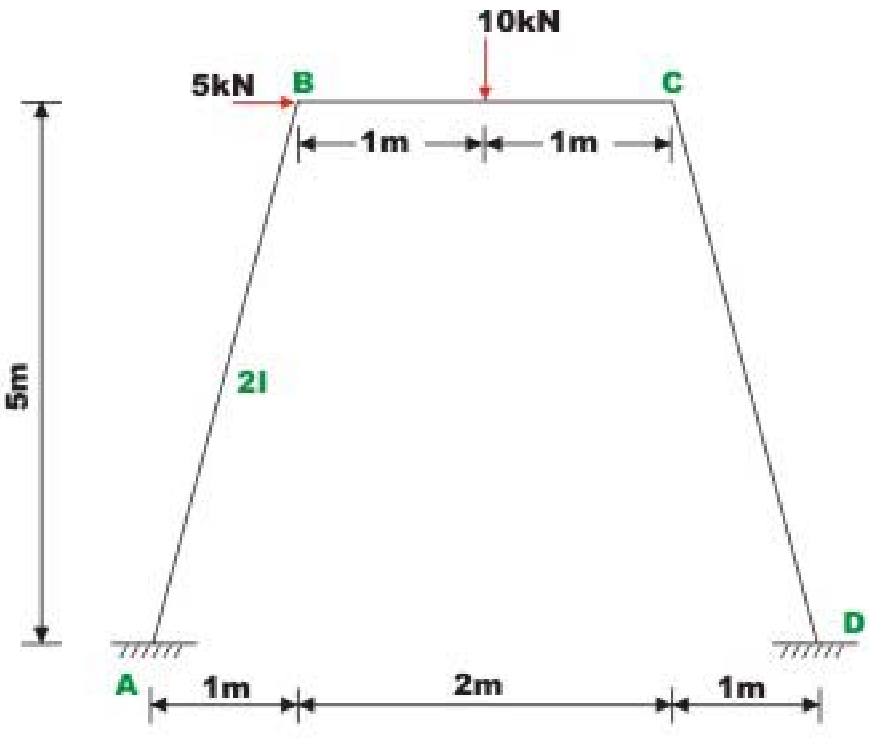

Fig.21.6a Example 21.3

Solution:

a) Calculate stiffness and distribution factors

$$K_{BA} = \frac{2EI}{5.1} = 0.392EI \;; \qquad K_{BC} = 0.50EI$$

$$K_{CB} = 0.50EI \;; \qquad K_{CD} = 0.392EI$$

At joint B :

$$\sum K = 0.892EI$$

$$DF_{BA} = 0.439 \;; \qquad DF_{BC} = 0.561$$

At joint C :

$$\sum K = 0.892EI$$

$$DF_{CB} = 0.561 \;; \qquad DF_{CD} = 0.439 \qquad (1)$$

b) Calculate fixed end moments due to applied loading.

$$M^F_{AB} = M^F_{BA} = M^F_{CD} = M^F_{DC} = 0 \quad \text{kN.m}$$

$$M^F_{BC} = 2.50 \quad \text{kN.m}$$

$$M^F_{CB} = -2.50 \quad \text{kN.m} \qquad (2)$$

c) Prevent sidesway by providing artificial support at C. Carry out moment-distribution for this case as shown in Fig. 21.6b.

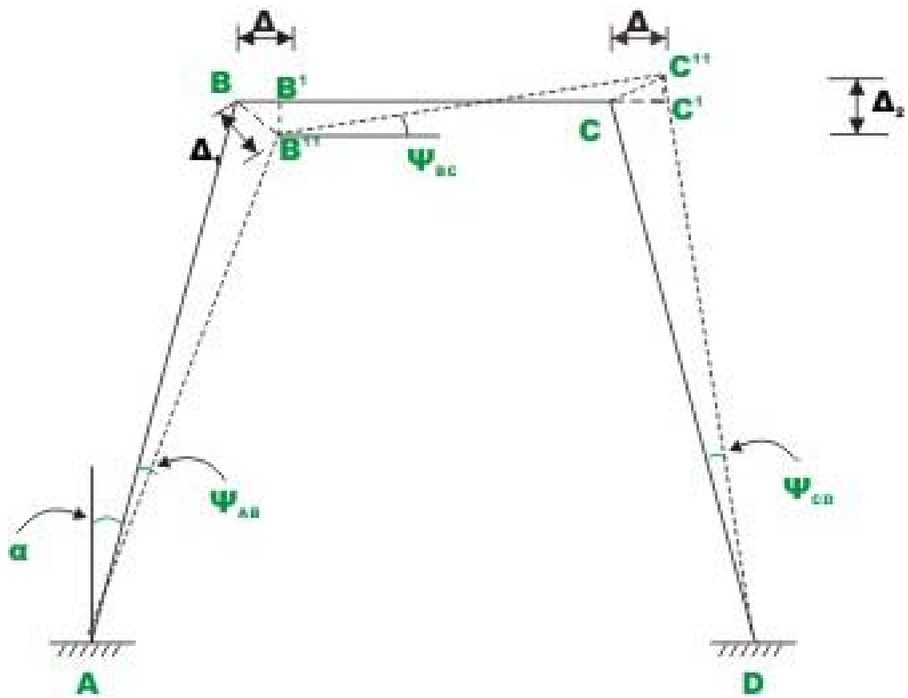

Fig.21.6b Rotation of Columns and beams

Now calculate reactions from free body diagram shown in Fig. 21.5d.

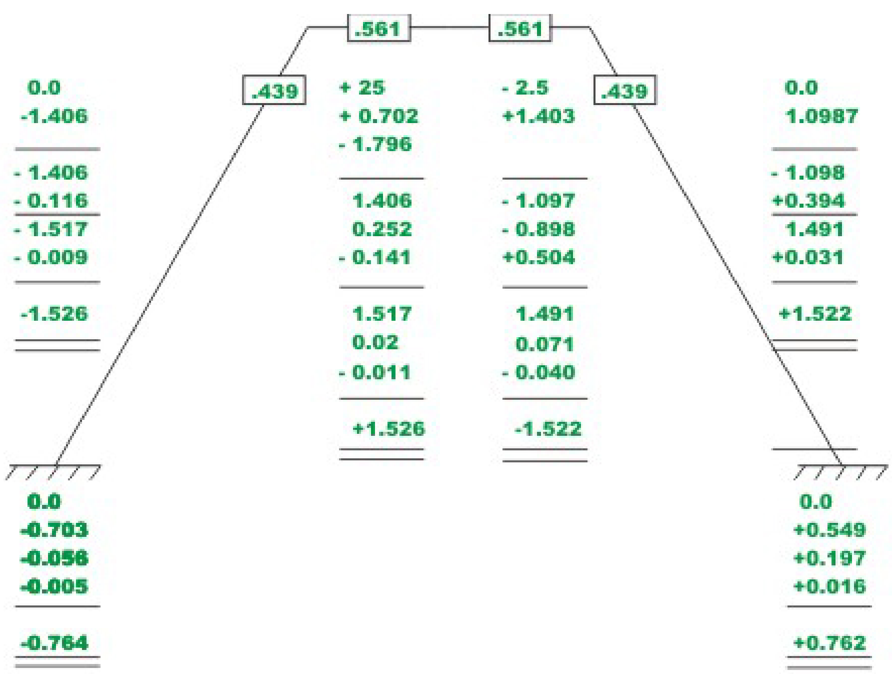

Fig. 21.6 © Moment distribution for applied loading

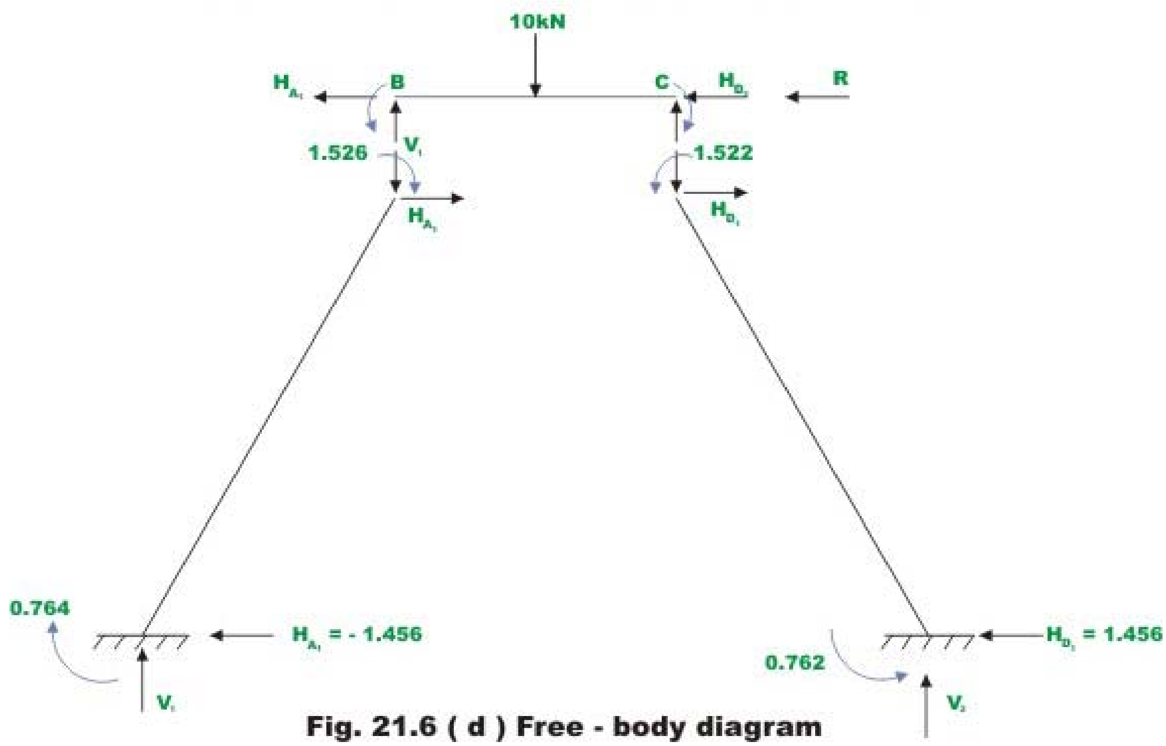

Fig. 21.6 (d) Free - body diagram

Column *AB*

$$\sum M_A = 0 \Rightarrow 5H_{A1} + 1.526 + 0.764 + V_1 = 0$$

$$5H_{A1} + V_1 = -2.29 \qquad (3)$$

Column *CD*

$$\sum M_D = 0 \Rightarrow 5H_{D1} - 1.522 - 0.762 - V_2 = 0$$

$$5H_{D1} - V_2 = 2.284 \qquad (4)$$

Beam *BC*

$$\sum M_C = 0 \Rightarrow 2V_1 + 1.522 - 1.526 - 10 \times 1 = 0$$

$$V_1 = 5.002 \quad \text{kN}(\uparrow)$$

$$V_2 = 4.998 \quad \text{kN}(\uparrow) \qquad (5)$$

Thus from (3) $H_{A1} = -1.458 \quad \text{kN}(\rightarrow)$

from (4) $H_{D1} = 1.456 \quad \text{kN}(\leftarrow)$ (6)

$$\sum F_X = 0 \qquad H_{A1} + H_{D1} + R - 5 = 0$$
$$R = +5.002 \ \ \text{kN} (\leftarrow) \qquad (7)$$

d) Moment-distribution for arbitrary sidesway Δ'.

Calculate fixed end beam moments for arbitrary sidesway of

$$\Delta' = \frac{12.75}{EI}$$

The member rotations for this arbitrary sidesway is shown in Fig. 21.6e.

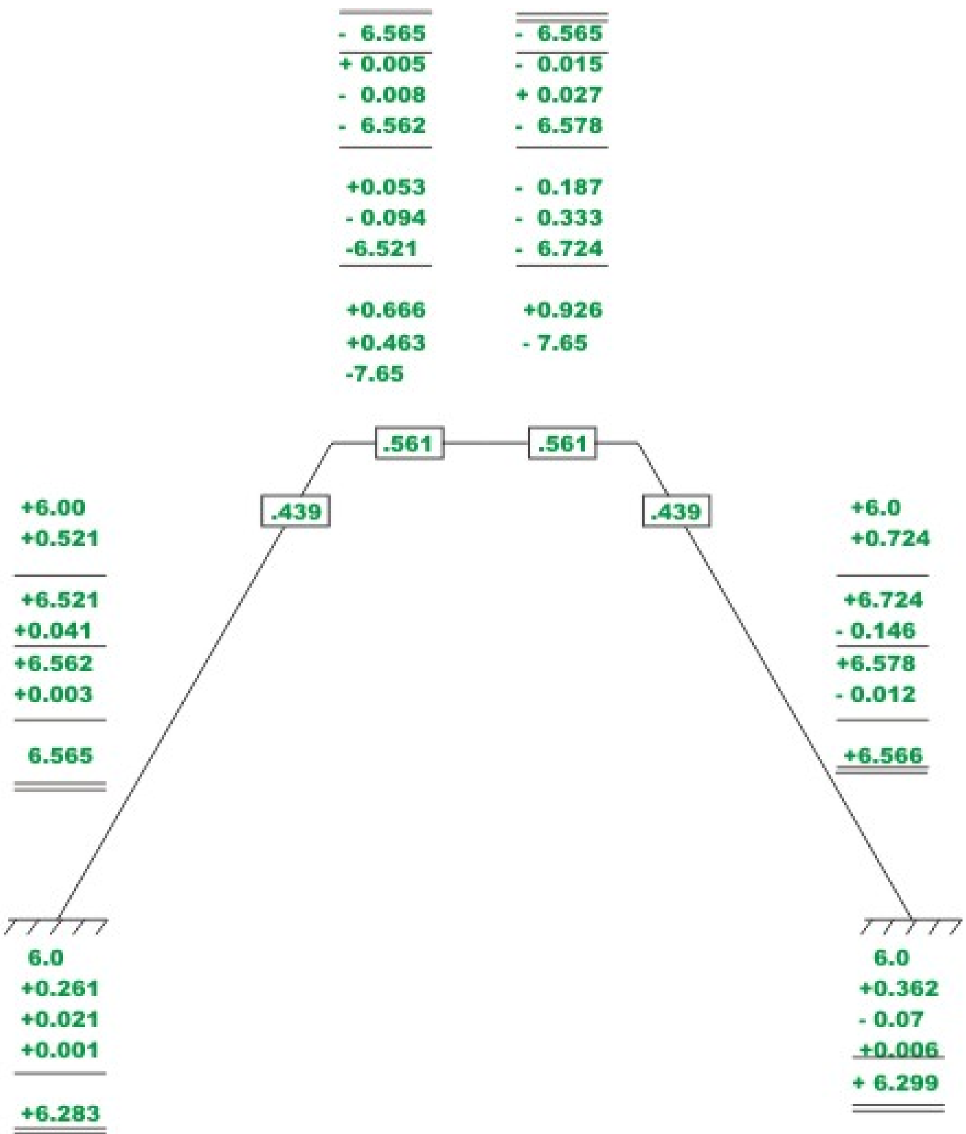

Fig. 21.6 (e) Moment distribution of arbitrary known sidesway

$$\psi_{AB} = \frac{BB''}{L_{AB}} = -\frac{\Delta_1}{L_{AB}} \quad ; \qquad \Delta_1 = \frac{\Delta'}{\cos\alpha} = \frac{5.1\Delta'}{5}$$

$$\Delta_2 = \frac{2\Delta'}{5} = 0.4\Delta'$$

$$\psi_{AB} = -\frac{\Delta'}{5}(clockwise)\,;\psi_{CD} = -\frac{\Delta'}{5}(clockwise)$$

$$\psi_{BC} = \frac{\Delta_2}{2} = \frac{2\Delta'\tan\alpha}{2} = \frac{\Delta'}{5}(counterclockwise)$$

$$M_{AB}^F = -\frac{6EI_{AB}}{L_{AB}}\psi_{AB} = -\frac{6E(2I)}{5.1}\left(-\frac{12.75}{5EI}\right) = +6.0 \text{ kN.m}$$

$$M_{BA}^F = +6.0 \quad \text{kN.m}$$

$$M_{BC}^F = -\frac{6EI_{BC}}{L_{BC}}\psi_{BC} = -\frac{6E(I)}{2}\left(\frac{12.75}{5EI}\right) = -7.65 \quad \text{kN.m}$$

$$M_{CB}^F = -7.65 \quad \text{kN.m}$$

$$M_{CD}^F = -\frac{6EI_{CD}}{L_{CD}}\psi_{CD} = -\frac{6E(2I)}{5.1}\left(-\frac{12.75}{5EI}\right) = +6.0 \quad \text{kN.m}$$

$$M_{DC}^F = +6.0 \quad \text{kN.m}$$

The moment-distribution for the arbitrary sway is shown in Fig. 21.6f. Now reactions can be calculated from statics.

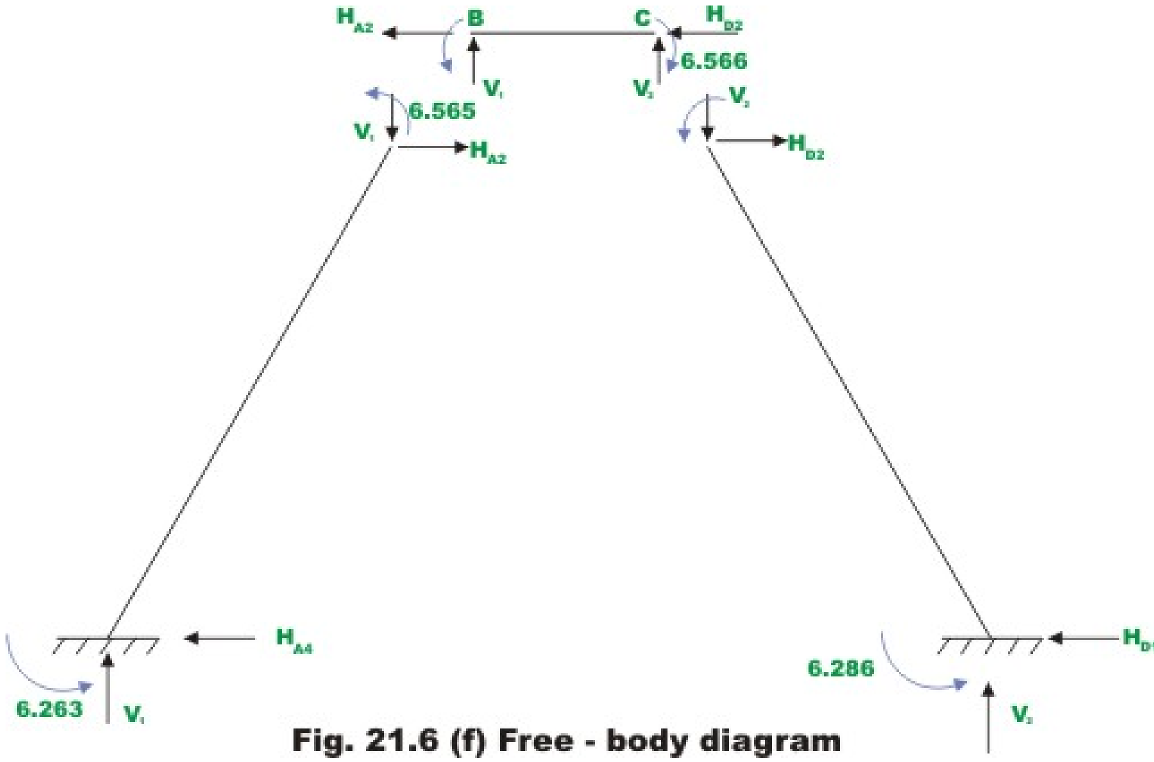

Fig. 21.6 (f) Free - body diagram

Column AB

$$\sum M_A = 0 \Rightarrow 5H_{A2} - 6.283 - 6.567 + V_1 = 0$$

$$5H_{A1} + V_1 = 12.85 \qquad (3)$$

Column CD

$$\sum M_D = 0 \Rightarrow 5H_{D2} - 6.567 - 6.283 - V_2 = 0$$

$$5H_{D1} - V_2 = 12.85 \qquad (4)$$

Beam BC

$$\sum M_C = 0 \Rightarrow 2V_1 + 6.567 + 6.567 = 0$$

$$V_1 = -6.567 \quad kN(\downarrow); V_2 = +6.567 \quad \text{kN}(\uparrow) \qquad (5)$$

Thus from 3 $H_{A2} = +3.883 \quad \text{kN}(\leftarrow)$

from 4 $H_{D2} = 3.883\ \text{kN}(\leftarrow)$ (6)

$$F = 7.766 \quad \text{kN}(\leftarrow) \qquad (7)$$

e) **Final results**

$$k\,F = R$$

$$k = \frac{5.002}{7.766} = 0.644$$

Now the actual end moments in the frame are,

$$M_{AB} = M'_{AB} + k\,M''_{AB}$$

$$M_{AB} = -0.764 + 0.644(+6.283) = +3.282 \quad \text{kN.m}$$

$$M_{BA} = -1.526 + 0.644(+6.567) = 2.703 \quad \text{kN.m}$$

$$M_{BC} = 1.526 + 0.644(-6.567) = -2.703 \quad \text{kN.m}$$

$$M_{CB} = -1.522 + 0.644(-6.567) = --5.751 \quad \text{kN.m}$$

$$M_{CD} = 1.522 + 0.644(6.567) = 5.751 \quad \text{kN.m}$$

$$M_{DC} = 0.762 + 0.644(6.283) = 4.808 \quad \text{kN.m}$$

The actual sway

$$\Delta = k\,\Delta' = 0.644 \times \frac{12.75}{EI}$$

$$= \frac{8.212}{EI}$$

Summary

In this lesson, the frames which are not restrained against sidesway are identified and solved by the moment-distribution method. The moment-distribution method is applied in two steps: in the first step, the frame prevented from sidesway but subjected to external loads is analysed and subsequently, the frame which is undergoing an arbitrary but known sidesway is analysed. Using shear equation for the frame, the moments in the frame is obtained. The numerical examples are explained with the help of free-body diagrams. The deflected shape of the frame is sketched to understand its deformation under external loads.