Module 3

Analysis of Statically Indeterminate Structures by the Displacement Method

Lesson
17
The Slope-Deflection Method: Frames with Sidesway

Instructional Objectives

After reading this chapter the student will be able to
1. Derive slope-deflection equations for the frames undergoing sidesway.
2. Analyse plane frames undergoing sidesway.
3, Draw shear force and bending moment diagrams.
4. Sketch deflected shape of the plane frame not restrained against sidesway.

17.1 Introduction

In this lesson, slope-deflection equations are applied to analyse statically indeterminate frames undergoing sidesway. As stated earlier, the axial deformation of beams and columns are small and are neglected in the analysis. In the previous lesson, it was observed that sidesway in a frame will not occur if

1. They are restrained against sidesway.
2. If the frame geometry and the loading are symmetrical.

In general loading will never be symmetrical. Hence one could not avoid sidesway in frames.

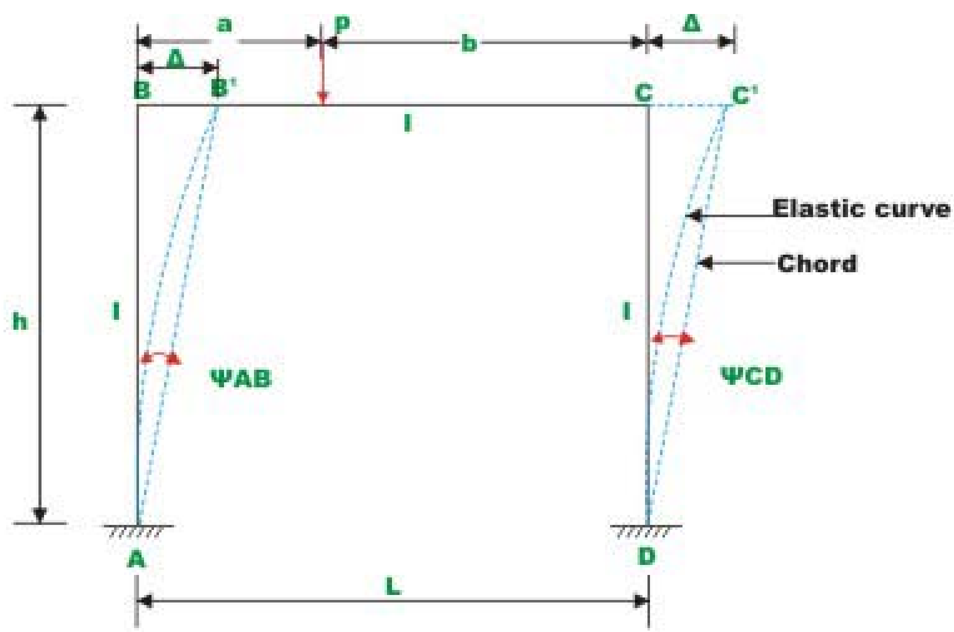

Fig.17.1 Plane frame undergoing sway

For example, consider the frame of Fig. 17.1. In this case the frame is symmetrical but not the loading. Due to unsymmetrical loading the beam end moments M_{BC} and M_{CB} are not equal. If b is greater than a, then $M_{BC} > M_{CB}$. In such a case joint B and C are displaced toward right as shown in the figure by an unknown amount Δ. Hence we have three unknown displacements in this frame: rotations θ_B, θ_C and the linear displacement Δ. The unknown joint rotations θ_B and θ_C are related to joint moments by the moment equilibrium equations. Similarly, when unknown linear displacement occurs, one needs to consider force-equilibrium equations. While applying slope-deflection equation to columns in the above frame, one must consider the column rotation $\psi\left(=\frac{\Delta}{h}\right)$ as unknowns. It is observed that in the column AB, the end B undergoes a linear displacement Δ with respect to end A. Hence the slope-deflection equation for column AB is similar to the one for beam undergoing support settlement. However, in this case Δ is unknown. For each of the members we can write the following slope-deflection equations.

$$M_{AB} = M_{AB}^F + \frac{2EI}{h}\left[2\theta_A + \theta_B - 3\psi_{AB}\right] \qquad \text{where } \psi_{AB} = -\frac{\Delta}{h}$$

ψ_{AB} is assumed to be negative as the chord to the elastic curve rotates in the clockwise directions.

$$\begin{aligned}
M_{BA} &= M_{BA}^F + \frac{2EI}{h}\left[2\theta_B + \theta_A - 3\psi_{AB}\right] \\
M_{BC} &= M_{BC}^F + \frac{2EI}{h}\left[2\theta_B + \theta_C\right] \\
M_{CB} &= M_{CB}^F + \frac{2EI}{h}\left[2\theta_C + \theta_B\right] \\
M_{CD} &= M_{CD}^F + \frac{2EI}{h}\left[2\theta_C + \theta_D - 3\psi_{CD}\right] \qquad \psi_{CD} = -\frac{\Delta}{h} \\
M_{DC} &= M_{DC}^F + \frac{2EI}{h}\left[2\theta_D + \theta_C - 3\psi_{CD}\right] \qquad (17.1)
\end{aligned}$$

As there are three unknowns (θ_B, θ_C and Δ), three equations are required to evaluate them. Two equations are obtained by considering the moment equilibrium of joint B and C respectively.

$$\sum M_B = 0 \quad \Rightarrow \quad M_{BA} + M_{BC} = 0 \qquad (17.2a)$$

$$\sum M_C = 0 \quad \Rightarrow \quad M_{CB} + M_{CD} = 0 \qquad (17.2b)$$

Now consider free body diagram of the frame as shown in Fig. 17.2. The horizontal shear force acting at A and B of the column AB is given by

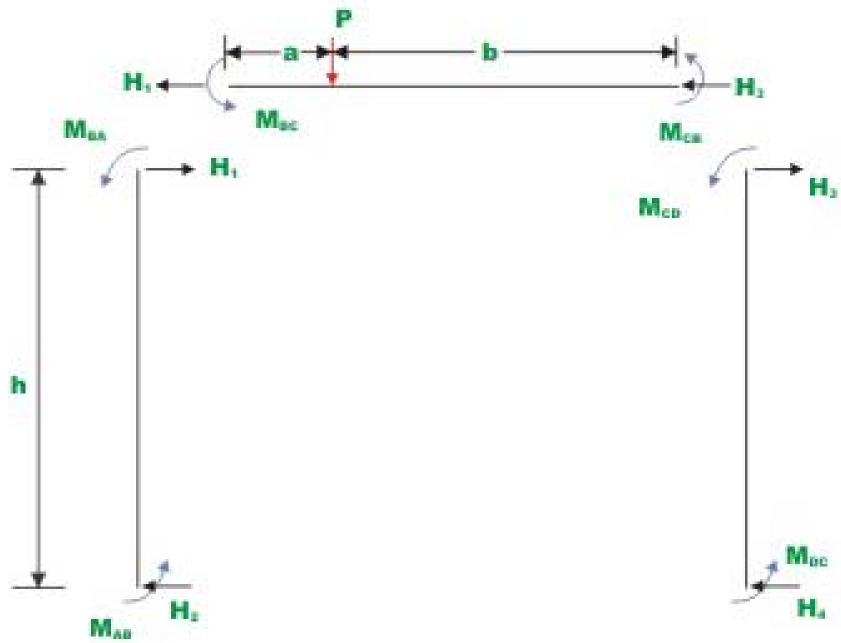

Fig.17.2 Free - body diagrams of columns and beams

$$H_1 = \frac{M_{BA} + M_{AB}}{h} \tag{17.3a}$$

Similarly for member CD, the shear force H_3 is given by

$$H_3 = \frac{M_{CD} + M_{DC}}{h} \tag{17.3b}$$

Now, the required third equation is obtained by considering the equilibrium of member BC,

$$\sum F_X = 0 \qquad \Rightarrow \qquad H_1 + H_3 = 0$$

$$\frac{M_{BA} + M_{AB}}{h} + \frac{M_{CD} + M_{DC}}{h} = 0 \tag{17.4}$$

Substituting the values of beam end moments from equation (17.1) in equations (17.2a), (17.2b) and (17.4), we get three simultaneous equations in three unknowns θ_B, θ_C and Δ, solving which joint rotations and translations are evaluated.

Knowing joint rotations and translations, beam end moments are calculated from slope-deflection equations. The complete procedure is explained with a few numerical examples.

Example 17.1

Analyse the rigid frame as shown in Fig. 17.3a. Assume EI to be constant for all members. Draw bending moment diagram and sketch qualitative elastic curve.

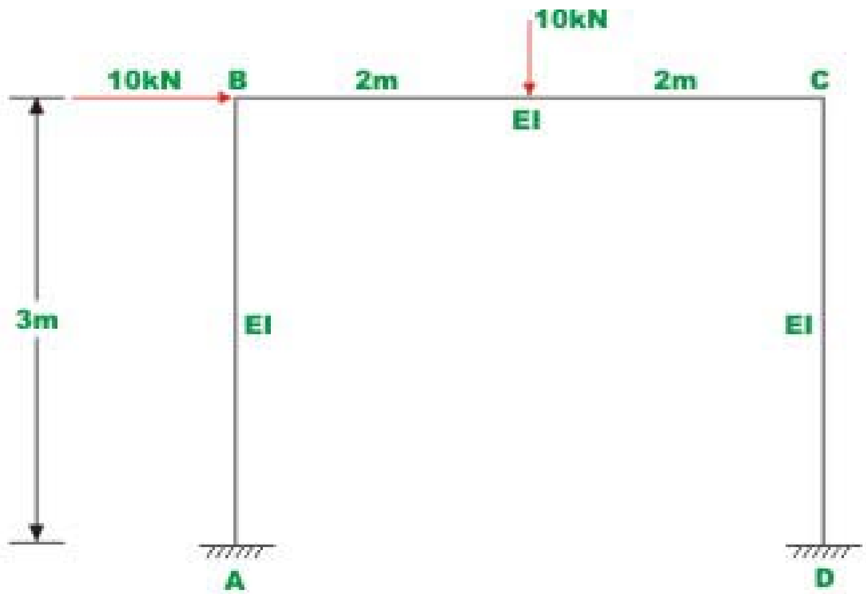

Fig.17.3 (a) Example 17.1

Solution

In the given problem, joints B and C rotate and also translate by an amount Δ. Hence, in this problem we have three unknown displacements (two rotations and one translation) to be evaluated. Considering the kinematically determinate structure, fixed end moments are evaluated. Thus,

$$M_{AB}^{F}=0\ ;M_{BA}^{F}=0\ ;M_{BC}^{F}=+10\,kN.m\ ;M_{CB}^{F}=-10kN.m\ ;M_{CD}^{F}=0\ ;M_{DC}^{F}=0. \qquad (1)$$

The ends A and D are fixed. Hence, $\theta_A=\theta_D=0$. Joints B and C translate by the same amount Δ. Hence, chord to the elastic curve AB' and DC' rotates by an amount (see Fig. 17.3b)

$$\psi_{AB}=\psi_{CD}=-\frac{\Delta}{3} \qquad (2)$$

Chords of the elastic curve AB' and DC' rotate in the clockwise direction; hence ψ_{AB} and ψ_{CD} are taken as negative.

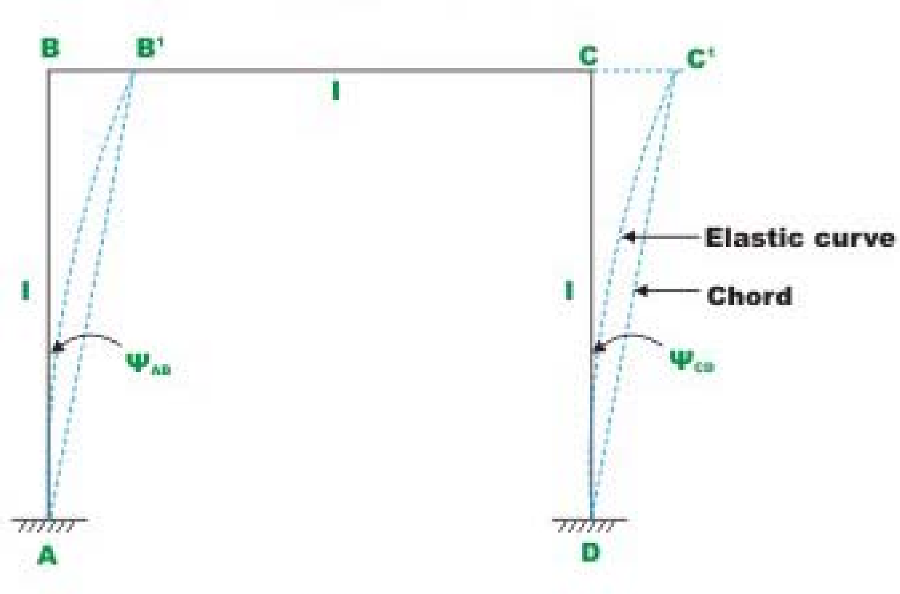

Fig.17.3b Column ratation

Now, writing the slope-deflection equations for the six beam end moments,

$$M_{AB} = M_{AB}^F + \frac{2EI}{3}\left[2\theta_A + \theta_B - 3\psi_{AB}\right]$$

$$M_{AB}^F = 0 \ ;\theta_A = 0 \ ;\psi_{AB} = -\frac{\Delta}{3}.$$

$$M_{AB} = \frac{2}{3}EI\theta_B + \frac{2}{3}EI\Delta$$

$$M_{BA} = \frac{4}{3}EI\theta_B + \frac{2}{3}EI\Delta$$

$$M_{BC} = 10 + EI\theta_B + \frac{1}{2}EI\theta_C$$

$$M_{CB} = -10 + \frac{1}{2}EI\theta_B + EI\theta_C$$

$$M_{CD} = \frac{4}{3}EI\theta_C + \frac{2}{3}EI\Delta$$

$$M_{DC} = \frac{2}{3}EI\theta_C + \frac{2}{3}EI\Delta \tag{3}$$

Now, consider the joint equilibrium of B and C (vide Fig. 17.3c).

$$\sum M_B = 0 \qquad \Rightarrow \qquad M_{BA} + M_{BC} = 0 \tag{4}$$

$$\sum M_C = 0 \qquad \Rightarrow \qquad M_{CB} + M_{CD} = 0 \tag{5}$$

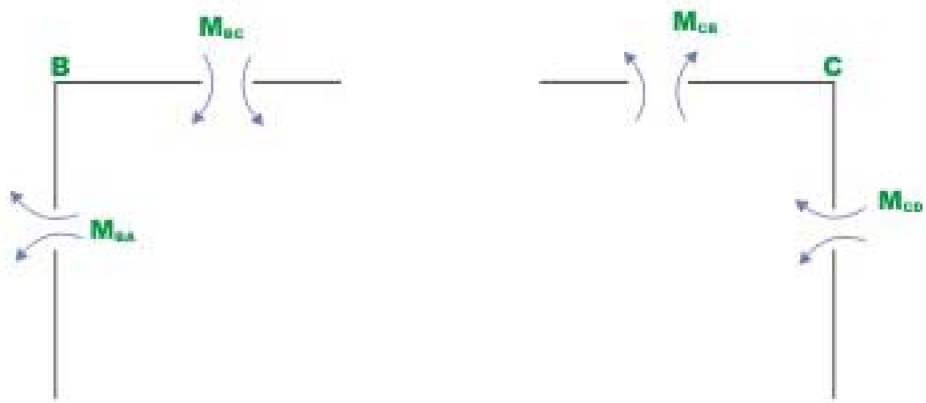

Fig.17.3c Free - body diagram of joints B and C

The required third equation is written considering the horizontal equilibrium of the entire frame *i.e.* $\sum F_X = 0$ (vide Fig. 17.3d).

$$-H_1 + 10 - H_2 = 0$$

$$\Rightarrow \qquad H_1 + H_2 = 10\,. \tag{6}$$

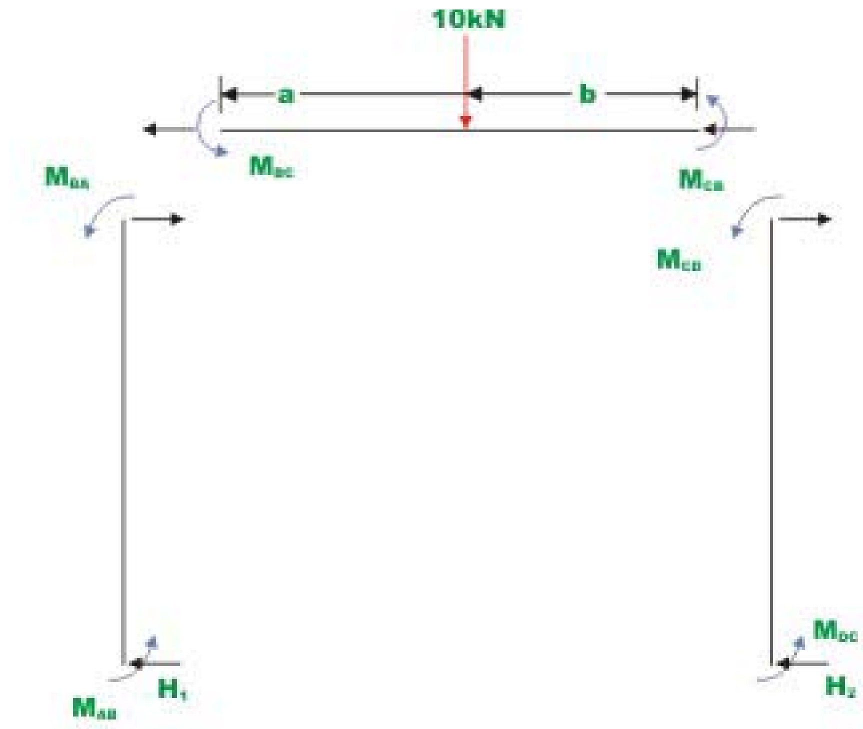

Fig.17.3d Free - body diagram of frame

Considering the equilibrium of the column AB and CD, yields

$$H_1 = \frac{M_{BA} + M_{AB}}{3}$$

and

$$H_2 = \frac{M_{CD} + M_{DC}}{3} \tag{7}$$

The equation (6) may be written as,

$$M_{BA} + M_{AB} + M_{CD} + M_{DC} = 30 \tag{8}$$

Substituting the beam end moments from equation (3) in equations (4), (5) and (6)

$$2.333EI\theta_B + 0.5EI\theta_C + 0.667EI\Delta = -10 \tag{9}$$

$$2.333EI\theta_C + 0.5EI\theta_B + 0.667EI\Delta = 10 \tag{10}$$

$$2EI\theta_B + 2EI\theta_C + \frac{8}{3}EI\Delta = 30 \qquad (11)$$

Equations (9), (10) and (11) indicate symmetry and this fact may be noted. This may be used as the check in deriving these equations.

Solving equations (9), (10) and (11),

$EI\theta_B = -9.572$; $EI\theta_C = 1.355$ and $EI\Delta = 17.417$.

Substituting the values of $EI\theta_B, EI\theta_C$ and $EI\Delta$ in the slope-deflection equation (3), one could calculate beam end moments. Thus,

$$M_{AB} = 5.23 \text{ kN.m} \quad \text{(counterclockwise)}$$

$$M_{BA} = -1.14 \text{ kN.m(clockwise)}$$

$$M_{BC} = 1.130 \text{ kN.m}$$

$$M_{CB} = -13.415 \text{ kN.m}$$

$$M_{CD} = 13.406 \text{ kN.m}$$

$$M_{DC} = 12.500 \text{ kN.m}.$$

The bending moment diagram for the frame is shown in Fig. 17.3 e. And the elastic curve is shown in Fig 17.3 f. the bending moment diagram is drawn on the compression side. Also note that the vertical hatching is used to represent bending moment diagram for the horizontal members (beams).

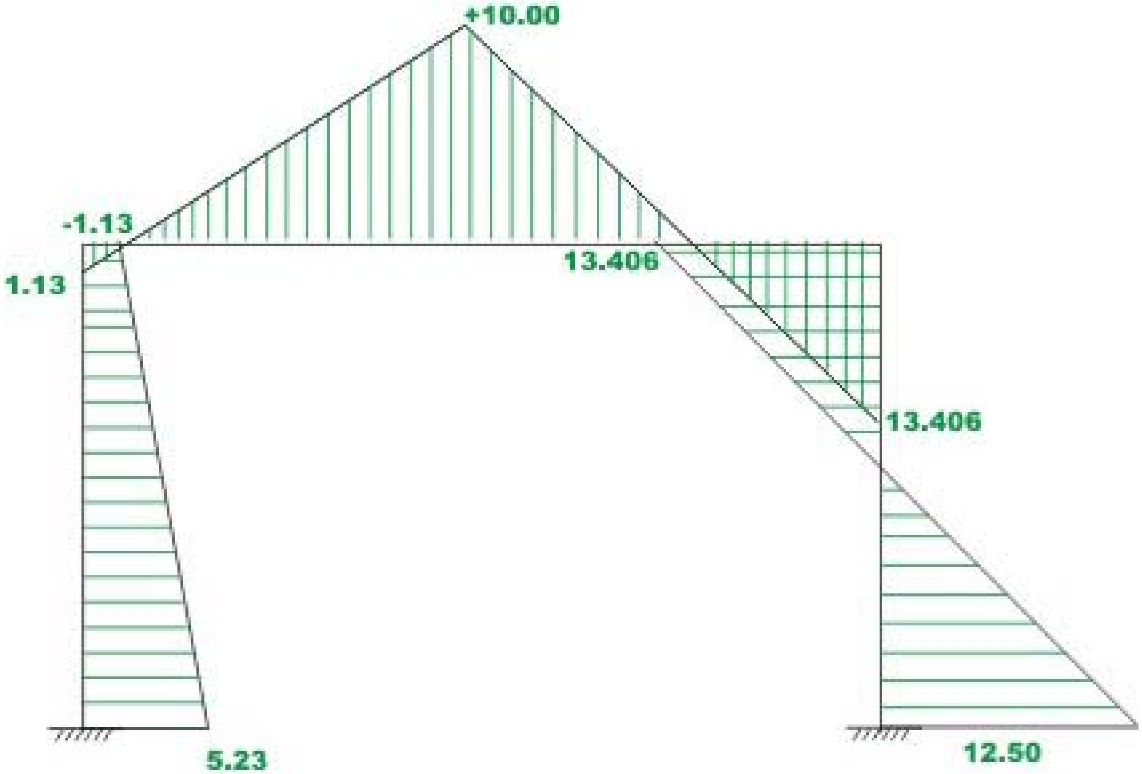

Fig.17.3e Bending moment diagram

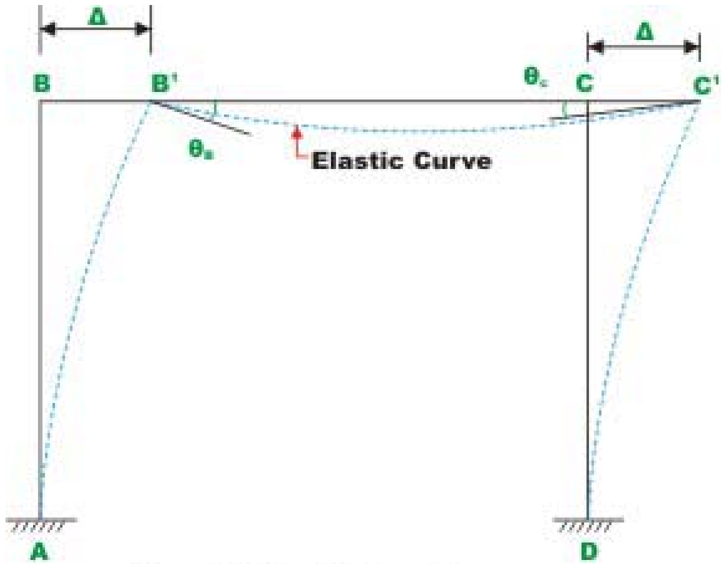

Fig.17.3f Elastic curve

Example 17.2

Analyse the rigid frame as shown in Fig. 17.4a and draw the bending moment diagram. The moment of inertia for all the members is shown in the figure. Neglect axial deformations.

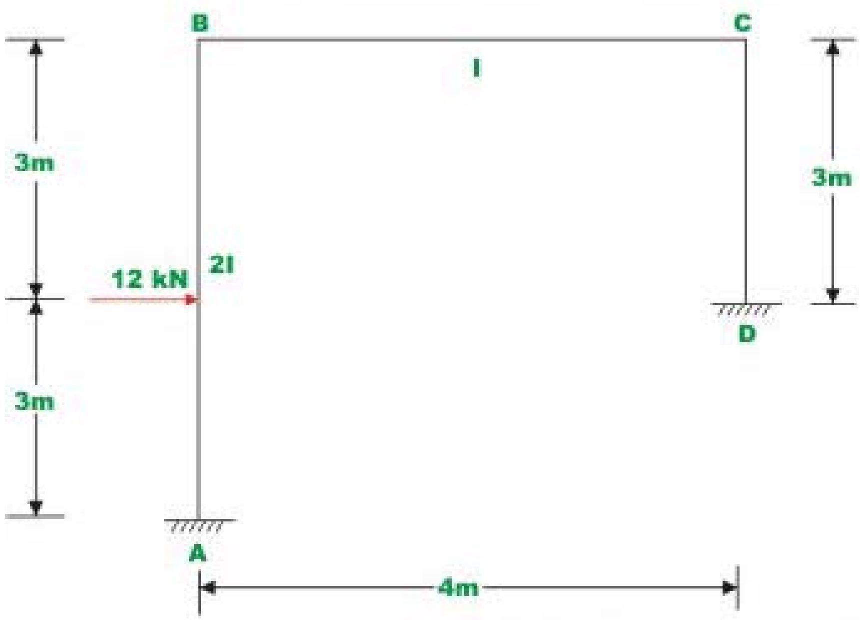

Fig.17.4a (Example 17.2)

Solution:

In this problem rotations and translations at joints B and C need to be evaluated. Hence, in this problem we have three unknown displacements: two rotations and one translation. Fixed end moments are

$$M_{AB}^{F}=\frac{12\times 3\times 9}{36}=9\,kN.m\;;M_{BA}^{F}=-9\,kN.m\;;$$
$$M_{BC}^{F}=0\;;M_{CB}^{F}=0\;;M_{CD}^{F}=0\;;M_{DC}^{F}=0. \qquad (1)$$

The joints B and C translate by the same amount Δ. Hence, the chord to the elastic curve rotates in the clockwise direction as shown in Fig. 17.3b.

$$\psi_{AB}=-\frac{\Delta}{6}$$

and
$$\psi_{CD}=-\frac{\Delta}{3} \qquad (2)$$

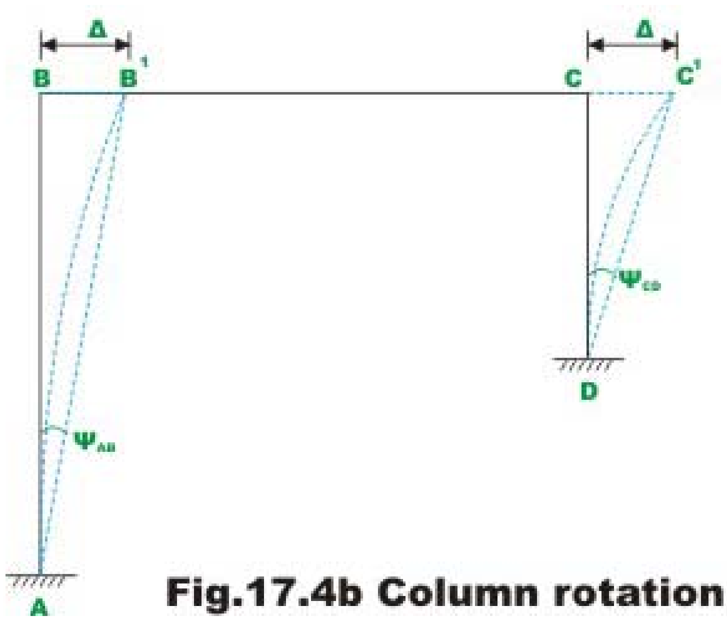

Fig.17.4b Column rotation due to sway

Now, writing the slope-deflection equations for six beam end moments,

$$M_{AB}=9+\frac{2(2EI)}{6}\left[\theta_B+\frac{\Delta}{2}\right]$$

$$M_{AB}=9+0.667EI\theta_B+0.333EI\Delta$$

$$M_{BA}=-9+1.333EI\theta_B+0.333EI\Delta$$

$$M_{BC} = EI\theta_B + 0.5EI\theta_C$$

$$M_{CB} = 0.5EI\theta_B + EI\theta_C$$

$$M_{CD} = 1.333EI\theta_C + 0.667EI\Delta$$

$$M_{DC} = 0.667EI\theta_C + 0.667EI\Delta \qquad (3)$$

Now, consider the joint equilibrium of B and C.

$$\sum M_B = 0 \qquad \Rightarrow \qquad M_{BA} + M_{BC} = 0 \qquad (4)$$

$$\sum M_C = 0 \qquad \Rightarrow \qquad M_{CB} + M_{CD} = 0 \qquad (5)$$

The required third equation is written considering the horizontal equilibrium of the entire frame. Considering the free body diagram of the member BC (vide Fig. 17.4c),

$H_1 + H_2 = 0$.

(6)

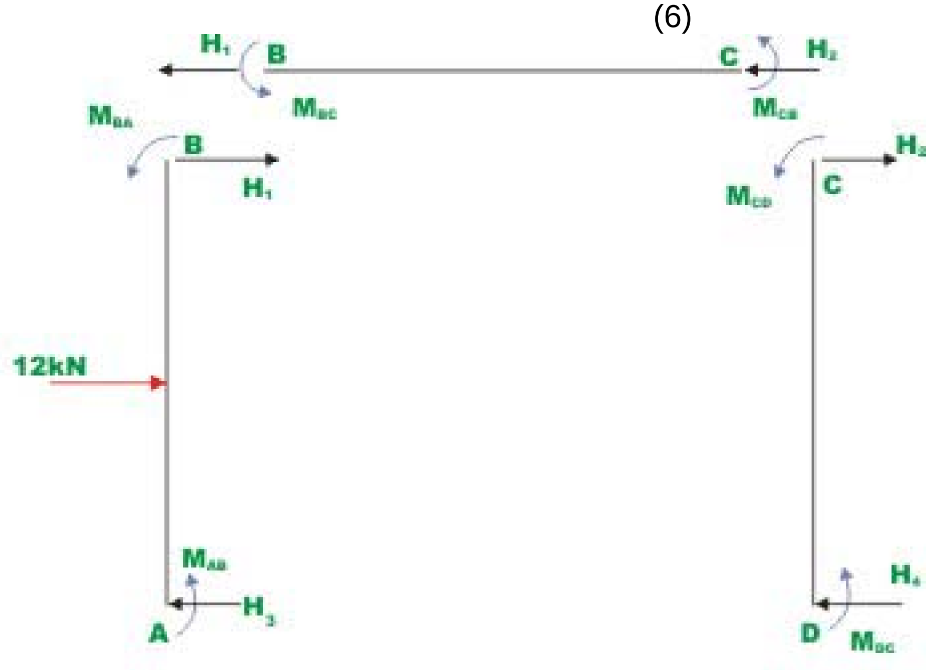

Fig.17.4c Free - body diagram

The forces H_1 and H_2 are calculated from the free body diagram of column AB and CD. Thus,

$$H_1 = -6 + \frac{M_{BA} + M_{AB}}{6}$$

and

$$H_2 = \frac{M_{CD} + M_{DC}}{3} \qquad (7)$$

Substituting the values of H_1 and H_2 into equation (6) yields,

$$M_{BA} + M_{AB} + 2M_{CD} + 2M_{DC} = 36 \qquad (8)$$

Substituting the beam end moments from equation (3) in equations (4), (5) and (8), yields

$$2.333EI\theta_B + 0.5EI\theta_C + 0.333EI\Delta = 9$$

$$2.333EI\theta_C + 0.5EI\theta_B + 0.667EI\Delta = 0$$

$$2EI\theta_B + 4EI\theta_C + 3.333EI\Delta = 36 \qquad (9)$$

Solving equations (9), (10) and (11),

$EI\theta_B = 2.76$; $EI\theta_C = -4.88$ and $EI\Delta = 15.00$.

Substituting the values of $EI\theta_B, EI\theta_C$ and $EI\Delta$ in the slope-deflection equation (3), one could calculate beam end moments. Thus,

$$M_{AB} = 15.835 \text{ kN.m} \quad \text{(counterclockwise)}$$
$$M_{BA} = -0.325 \text{ kN.m(clockwise)}$$
$$M_{BC} = 0.32 \text{ kN.m}$$
$$M_{CB} = -3.50 \text{ kN.m}$$
$$M_{CD} = 3.50 \text{ kN.m}$$
$$M_{DC} = 6.75 \text{ kN.m}.$$

The bending moment diagram for the frame is shown in Fig. 17.4 d.

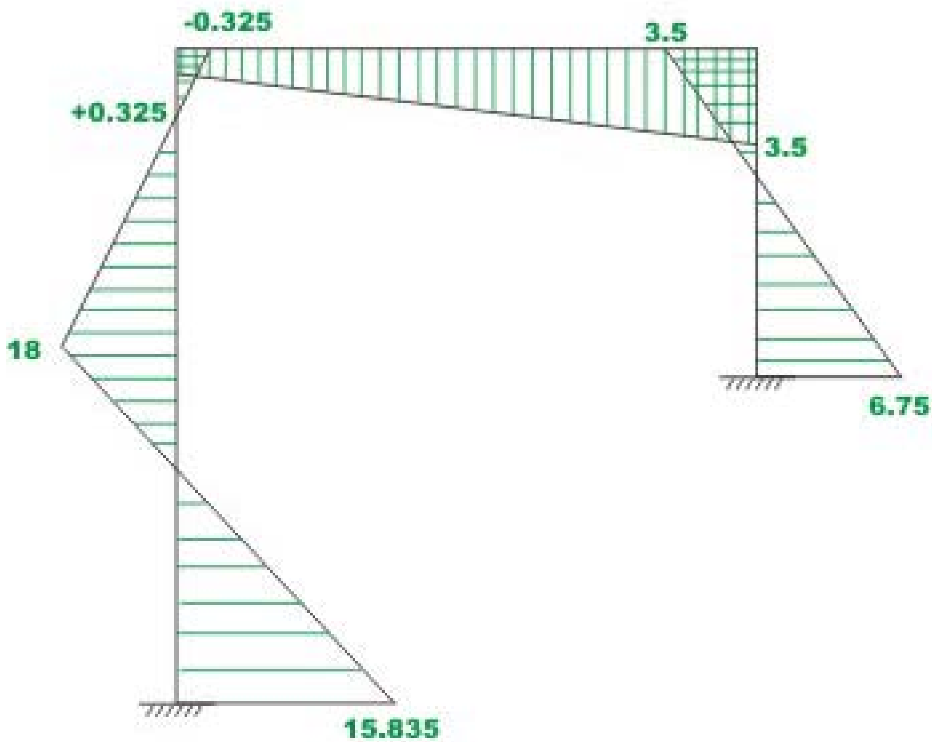

Fig.17.4d Bending moment diagram

Example 17.3

Analyse the rigid frame shown in Fig. 17.5 a. Moment of inertia of all the members are shown in the figure. Draw bending moment diagram.

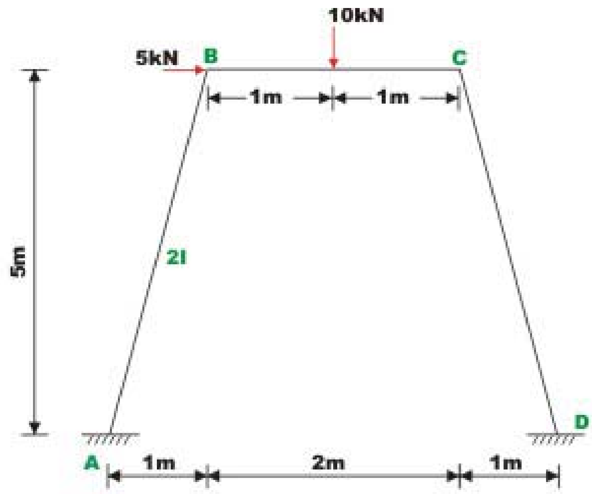

Fig.17.5a Example 17.3

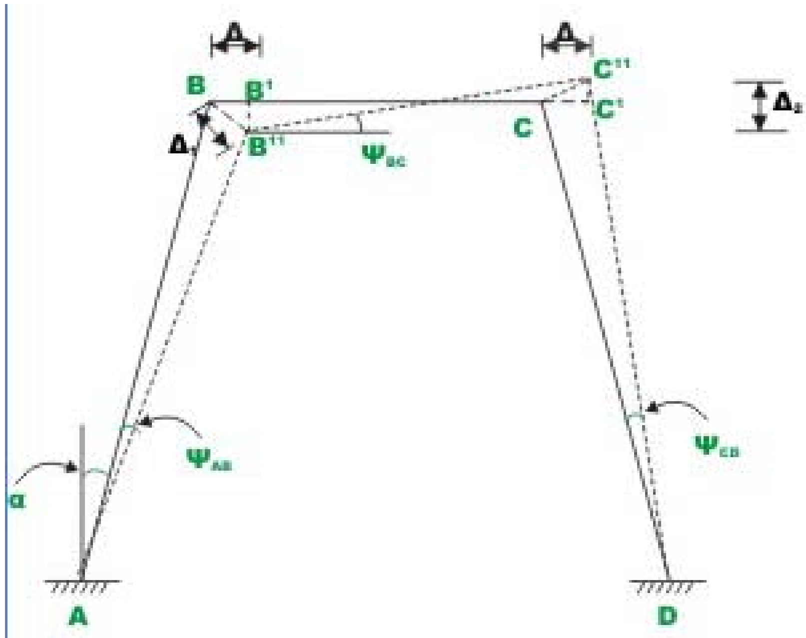

Fig.17.5b Rotation of Columns and beams

Under the action of external forces, the frame gets deformed as shown in Fig. 17.5b. In this figure, chord to the elastic curve are shown by dotted line. BB' is perpendicular to AB and CC'' is perpendicular to DC. The chords to the elastic curve AB'' rotates by an angle ψ_{AB}, $B''C''$ rotates by ψ_{BC} and DC rotates by ψ_{CD} as shown in figure. Due to symmetry, $\psi_{CD} = \psi_{AB}$. From the geometry of the figure,

$$\psi_{AB} = \frac{BB''}{L_{AB}} = -\frac{\Delta_1}{L_{AB}}$$

But

$$\Delta_1 = \frac{\Delta}{\cos\alpha}$$

Thus,

$$\psi_{AB} = -\frac{\Delta}{L_{AB}\cos\alpha} = -\frac{\Delta}{5}$$

$$\psi_{CD} = -\frac{\Delta}{5}$$

$$\psi_{BC} = \frac{\Delta_2}{2} = \frac{2\Delta\tan\alpha}{2} = \Delta\tan\alpha = \frac{\Delta}{5} \qquad (1)$$

We have three independent unknowns for this problem θ_B, θ_C and Δ. The ends A and D are fixed. Hence, $\theta_A = \theta_D = 0$. Fixed end moments are,

$$M_{AB}^F = 0 \;;M_{BA}^F = 0 \;;M_{BC}^F = +2.50\,kN.m \;;M_{CB}^F = -2.50kN.m \;;M_{CD}^F = 0 \;;M_{DC}^F = 0.$$

Now, writing the slope-deflection equations for the six beam end moments,

$$M_{AB} = \frac{2E(2I)}{5.1}\left[\theta_A - 3\psi_{AB}\right]$$

$$M_{AB} = 0.784EI\theta_B + 0.471EI\Delta$$
$$M_{BA} = 1.568EI\theta_B + 0.471EI\Delta$$

$$M_{BC} = 2.5 + 2EI\theta_B + EI\theta_C - 0.6EI\Delta$$
$$M_{BC} = -2.5 + EI\theta_B + 2EI\theta_C - 0.6EI\Delta$$

$$M_{CD} = 1.568EI\theta_C + 0.471EI\Delta$$
$$M_{DC} = 0.784EI\theta_C + 0.471EI\Delta \qquad (2)$$

Now, considering the joint equilibrium of B and C, yields

$$\sum M_B = 0 \qquad \Rightarrow \qquad M_{BA} + M_{BC} = 0$$

$$3.568EI\theta_B + EI\theta_C - 0.129EI\Delta = -2.5 \tag{3}$$

$$\sum M_C = 0 \qquad \Rightarrow \qquad M_{CB} + M_{CD} = 0$$

$$3.568EI\theta_C + EI\theta_B - 0.129EI\Delta = 2.5 \tag{4}$$

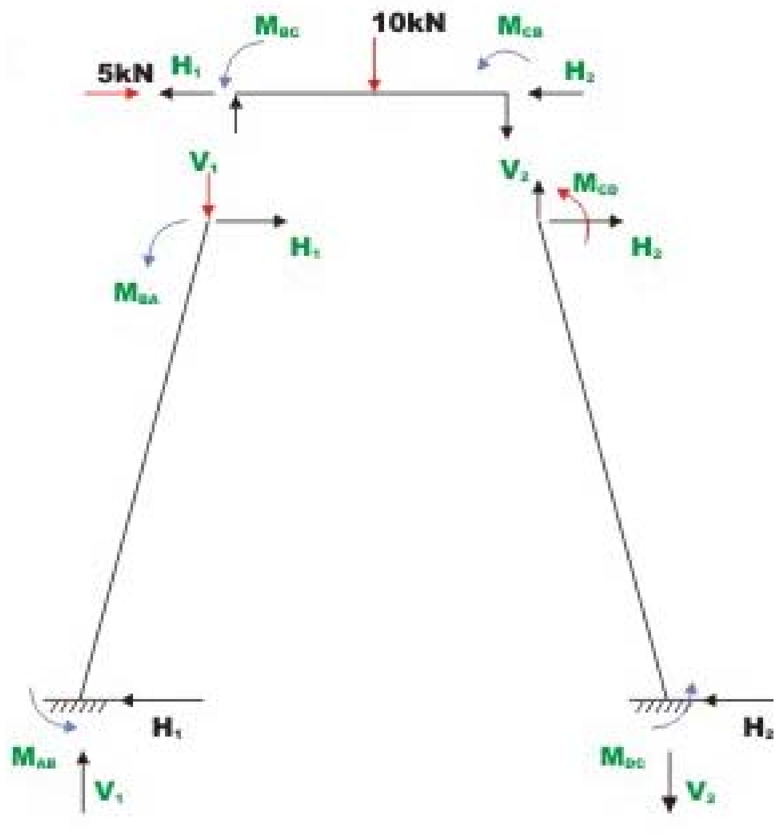

Fig.17.5c Free- body diagram

Shear equation for
Column AB

$$5H_1 - M_{AB} - M_{BA} + (1)V_1 = 0 \tag{5}$$

Column CD

$$5H_2 - M_{CD} - M_{DC} + (1)V_2 = 0 \tag{6}$$

Beam BC

$$\sum M_C = 0 \qquad 2V_1 - M_{BC} - M_{CB} - 10 = 0 \tag{7}$$

$$\sum F_X = 0 \qquad H_1 + H_2 = 5 \tag{8}$$

$$\sum F_Y = 0 \qquad V_1 - V_2 - 10 = 0 \tag{9}$$

From equation (7), $V_1 = \dfrac{M_{BC} + M_{CB} + 10}{2}$

From equation (8), $H_1 = 5 - H_2$

From equation (9), $V_2 = V_1 - 10 = \dfrac{M_{BC} + M_{CB} + 10}{2} - 10$

Substituting the values of V_1, H_1 and V_2 in equations (5) and (6),

$$60 - 10H_2 - 2M_{AB} - 2M_{BA} + M_{BC} + M_{CB} = 0 \tag{10}$$

$$-10 + 10H_2 - 2M_{CD} - 2M_{DC} + M_{BC} + M_{CB} = 0 \tag{11}$$

Eliminating H_2 in equation (10) and (11),

$$M_{AB} + M_{BA} + M_{CD} + M_{DC} - M_{BC} - M_{CB} = 25 \tag{12}$$

Substituting the values of $M_{AB}, M_{BA}, M_{CD}, M_{DC}$ in (12) we get the required third equation. Thus,

$$0.784EI\theta_B + 0.471EI\Delta + 1.568EI\theta_B + 0.471EI\Delta + 1.568EI\theta_C + 0.471EI\Delta +$$
$$0.784EI\theta_C + 0.471EI\Delta \text{-}(2.5 + 2EI\theta_B + EI\theta_C - 0.6EI\Delta)\text{-}$$
$$(-2.5 + EI\theta_B + 2EI\theta_C - 0.6EI\Delta) = 25$$

Simplifying,

$$-0.648EI\theta_C - 0.648EI\theta_B + 3.084EI\Delta = 25 \tag{13}$$

Solving simultaneously equations (3) (4) and (13), yields

$EI\theta_B = -0.741$; $EI\theta_C = 1.205$ and $EI\Delta = 8.204$.

Substituting the values of $EI\theta_B, EI\theta_C$ and $EI\Delta$ in the slope-deflection equation (3), one could calculate beam end moments. Thus,

$$M_{AB} = 3.28 \text{ kN.m}$$

$$M_{BA} = 2.70 \text{ kN.m}$$

$$M_{BC} = -2.70 \text{ kN.m}$$

$$M_{CB} = -5.75 \text{ kN.m}$$

$$M_{CD} = 5.75 \text{ kN.m}$$

$$M_{DC} = 4.81 \text{ kN.m}. \quad (14)$$

The bending moment diagram for the frame is shown in Fig. 17.5 d.

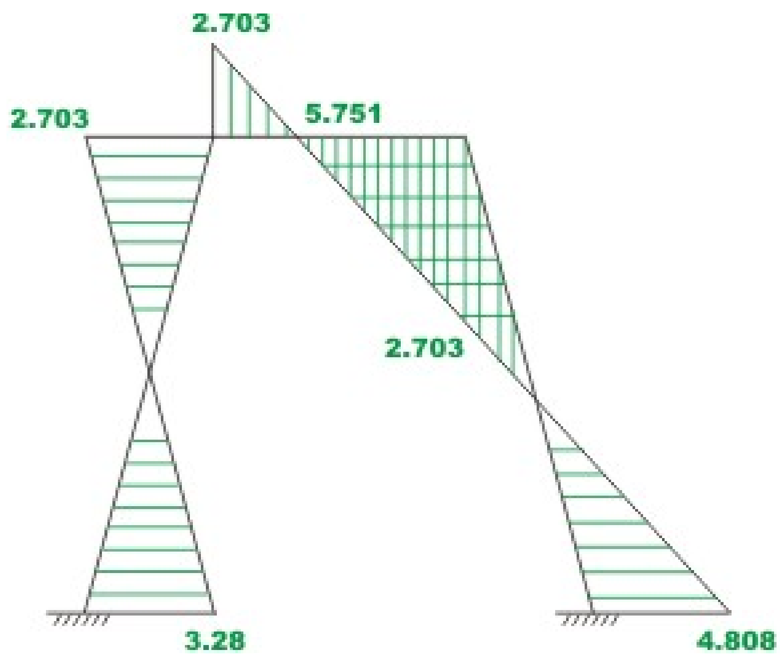

Fig.17.5d Bending moment diagram

Summary

In this lesson, slope-deflection equations are derived for the plane frame undergoing sidesway. Using these equations, plane frames with sidesway are analysed. The reactions are calculated from static equilibrium equations. A couple of problems are solved to make things clear. In each numerical example, the bending moment diagram is drawn and deflected shape is sketched for the plane frame.