Module
3
Analysis of Statically Indeterminate Structures by the Displacement Method

Lesson
15
The Slope-Deflection Method: Beams (Continued)

Instructional Objectives

After reading this chapter the student will be able to

1. Derive slope-deflection equations for the case beam with yielding supports.
2. Estimate the reactions induced in the beam due to support settlements.
3. Analyse the beam undergoing support settlements and subjected to external loads.
4. Write joint equilibrium equations in terms of moments.
5. Relate moments to joint rotations and support settlements.

15.1 Introduction

In the last lesson, slope-deflection equations were derived without considering the rotation of the beam axis. In this lesson, slope-deflection equations are derived considering the rotation of beam axis. In statically indeterminate structures, the beam axis rotates due to support yielding and this would in turn induce reactions and stresses in the structure. Hence, in this case the beam end moments are related to rotations, applied loads and beam axes rotation. After deriving the slope-deflection equation in section 15.2, few problems are solved to illustrate the procedure.

Consider a beam AB as shown in Fig.15.1.The support B is at a higher elevation compared to A by an amount Δ. Hence, the member axis has rotated by an amount ψ from the original direction as shown in the figure. Let L be the span of the beam and flexural rigidity of the beam EI, is assumed to be constant for the beam. The chord has rotated in the counterclockwise direction with respect to its original direction. The counterclockwise moment and rotations are assumed to be positive. As stated earlier, the slopes and rotations are derived by superposing the end moments developed due to

(1) Externally applied moments on beams.
(2) Displacements θ_A, θ_B and Δ (settlement)

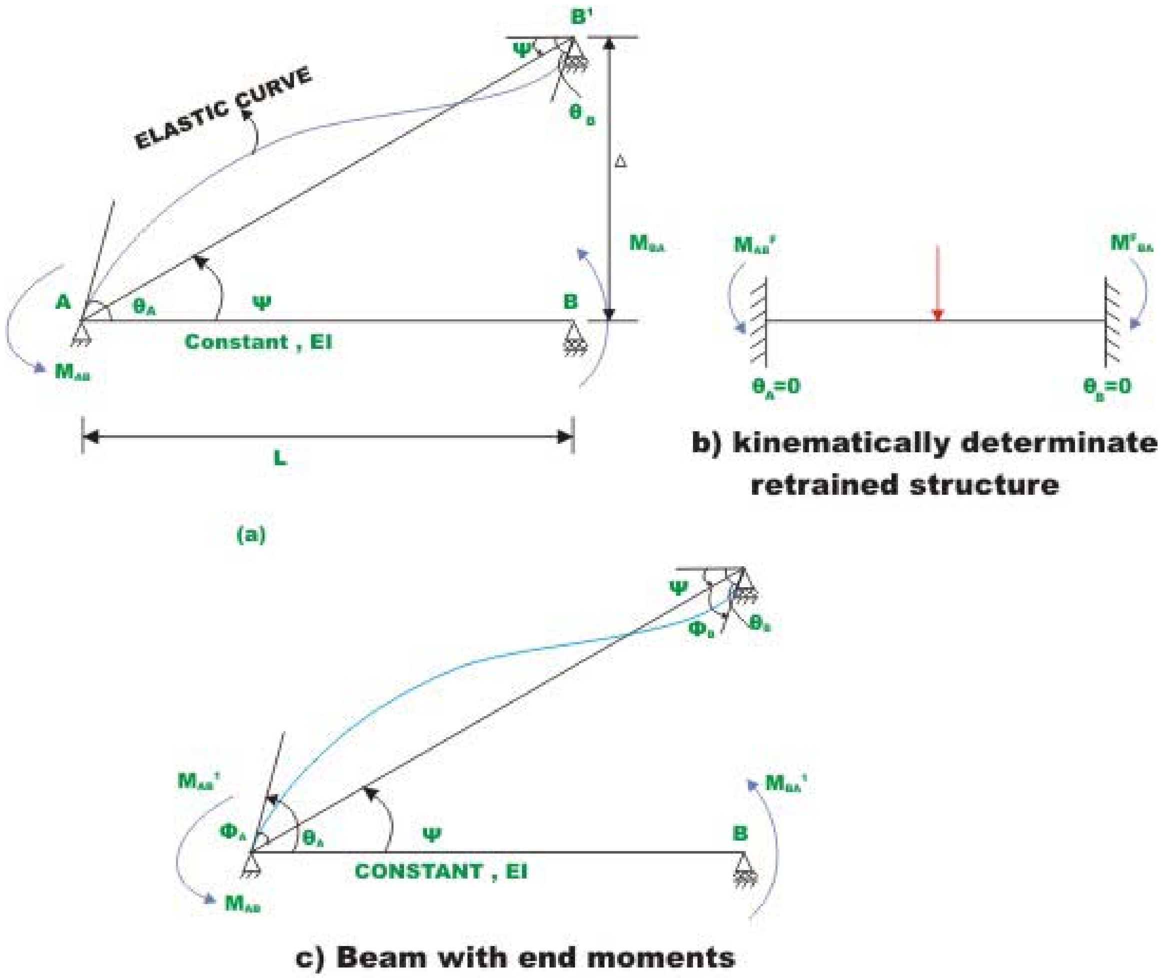


Figure 15.1

The given beam with initial support settlement may be thought of as superposition of two simple cases as shown in Fig.15.1 (b) and in Fig. 15.1(c). In Fig.15.1b, the kinematically determinate beam is shown with the applied load. For this case, the fixed end moments are calculated by force method. Let ϕ_A and ϕ_B be the end rotations of the elastic curve with respect to rotated beam axis AB' (see Fig.15.1c) that are caused by end moments $M_{AB}^{'}$ and $M_{BA}^{'}$. Assuming that rotations and displacements shown in Fig.15.1c are so small that

$$\tan\psi = \psi = \frac{\Delta}{l} \qquad (15.1)$$

Also, using the moment area theorem, ϕ_A and ϕ_B are written as

$$\phi_A = \theta_A - \psi = \frac{M_{AB}'L}{3EI} - \frac{M_{AB}'L}{6EI} \qquad (15.2a)$$

$$\phi_B = \theta_B - \psi = \frac{M_{BA}'L}{3EI} - \frac{M_{AB}'L}{6EI} \quad (15.2b)$$

Now solving for $M_A^{'}$ and $M_B^{'}$ in terms of θ_A, θ_B and ψ ,

$$M_{AB}' = \frac{2EI}{L}(2\theta_A + \theta_B - 3\psi) \quad (15.3a)$$

$$M_{BA}' = \frac{2EI}{L}(2\theta_B + \theta_A - 3\psi) \quad (15.3b)$$

Now superposing the fixed end moments due to external load and end moments due to displacements, the end moments in the actual structure is obtained .Thus (see Fig.15.1)

$$M_{AB} = M_{AB}^{F} + M_{AB}' \quad (15.4a)$$

$$M_{BA} = M_{BA}^{F} + M_{BA}' \quad (15.4b)$$

Substituting for $M_{AB}^{'}$ and $M_{BA}^{'}$ in equation (15.4a) and (15.4b), the slope-deflection equations for the general case are obtained. Thus,

$$M_{AB} = M_{AB}^{F} + \frac{2EI}{L}(2\theta_A + \theta_B - 3\psi) \quad (15.5a)$$

$$M_{BA} = M_{BA}^{F} + \frac{2EI}{L}(2\theta_B + \theta_A - 3\psi) \quad (15.5b)$$

In the above equations, it is important to adopt consistent sign convention. In the above derivation Δ is taken to be negative for downward displacements.

Example 15.1

Calculate the support moments in the continuous beam ABC (see Fig.15.2a) having constant flexural rigidity EI throughout ,due to vertical settlement of the support B by 5mm. Assume E =200 GPa and $I = 4 \times 10^{-4} m^4$ .Also plot quantitative elastic curve.

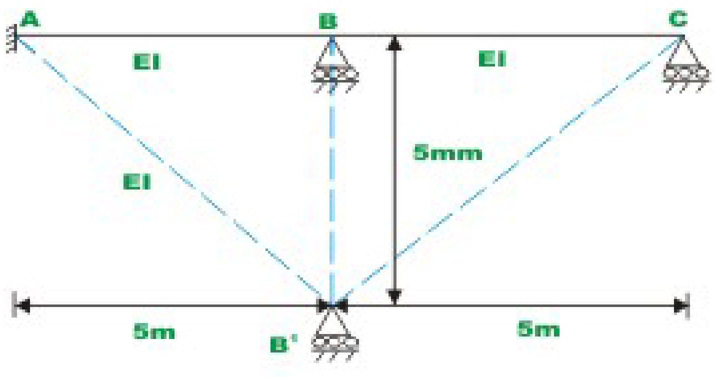


Figure 15.2 (a)

In the continuous beam ABC, two rotations θ_B and θ_C need to be evaluated. Hence, beam is kinematically indeterminate to second degree. As there is no external load on the beam, the fixed end moments in the restrained beam are zero (see Fig.15.2b).

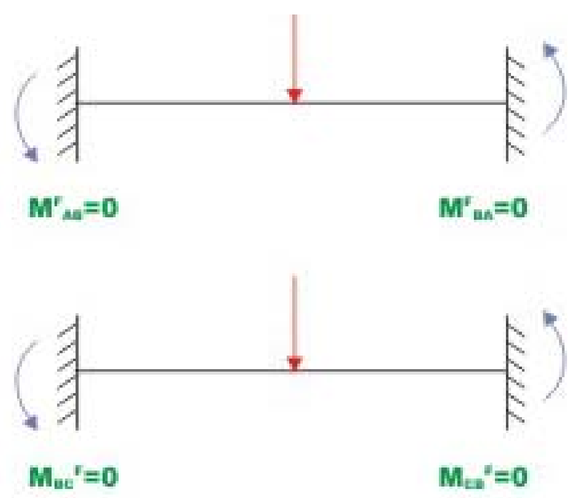


Figure 15.2 (b)

For each span, two slope-deflection equations need to be written. In span AB, B is below A. Hence, the chord AB rotates in clockwise direction. Thus, ψ_{AB} is taken as negative.

$$\psi_{AB} = \frac{-5\times10^{-3}}{5} = -1\times10^{-3} \quad (1)$$

Writing slope-deflection equation for span AB ,

$$M_{AB} = \frac{2EI}{L}\left(2\theta_A + \theta_B - 3\psi_{AB}\right)$$

For span AB , $\theta_A = 0$, Hence,

$$M_{AB} = \frac{2EI}{5}\left(\theta_B + 3\times10^{-3}\right)$$

$$M_{AB} = O.4EI\theta_B + .0012EI \quad (2)$$

Similarly, for beam-end moment at end B , in span AB

$$M_{BA} = 0.4EI\left(2\theta_B + 3\times10^{-3}\right)$$
$$M_{BA} = 0.8EI\theta_B + 0.0012EI \quad (3)$$

In span BC , the support C is above support B , Hence the chord joining $B'C$ rotates in anticlockwise direction.

$$\psi_{BC} = \psi_{CB} = 1\times10^{-3} \quad (4)$$

Writing slope-deflection equations for span BC ,

$$M_{BC} = 0.8EI\theta_B + 0.4EI\theta_C - 1.2\times10^{-3}EI$$

$$M_{CB} = 0.8EI\theta_C + 0.4EI\theta_B - 1.2\times10^{-3}EI \quad (5)$$

Now, consider the joint equilibrium of support B (see Fig.15.2c)

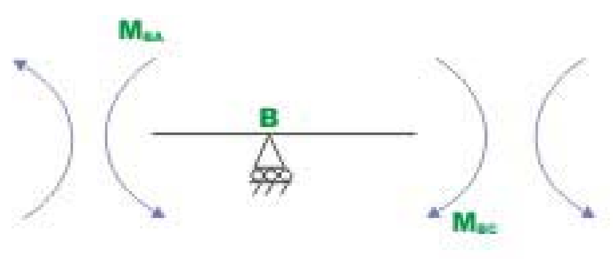


Fig 15.2c Free body diagram of joint B

$$M_{BA} + M_{BC} = 0 \qquad (6)$$

Substituting the values of M_{BA} and M_{BC} in equation (6),

$$0.8EI\theta_B + 1.2\times10^{-3}EI + 0.8EI\theta_B + 0.4EI\theta_C - 1.2\times10^{-3}EI = 0$$

Simplifying,

$$1.6\theta_B + 0.4\theta_C = 1.2\times10^{-3} \qquad (7)$$

Also, the support C is simply supported and hence, $M_{CB} = 0$

$$M_{CB} = 0 = 0.8\theta_C + 0.4\theta_B - 1.2\times10^{-3}EI$$

$$0.8\theta_C + 0.4\theta_B = 1.2\times10^{-3} \qquad (8)$$

We have two unknowns θ_B and θ_C and there are two equations in θ_B and θ_C. Solving equations (7) and (8)

$$\theta_B = -0.4286\times10^{-3} \text{ radians}$$

$$\theta_C = 1.7143\times10^{-3} \text{ radians} \qquad (9)$$

Substituting the values of θ_B, θ_C and EI in slope-deflection equations,

$$M_{AB} = 82.285 \text{ kN.m}$$

$$M_{BA} = 68.570 \text{ kN.m}$$

$$M_{BC} = -68.573 \text{ kN.m}$$

$$M_{CB} = 0 \text{ kN.m} \qquad (10)$$

Reactions are obtained from equations of static equilibrium (vide Fig.15.2d)

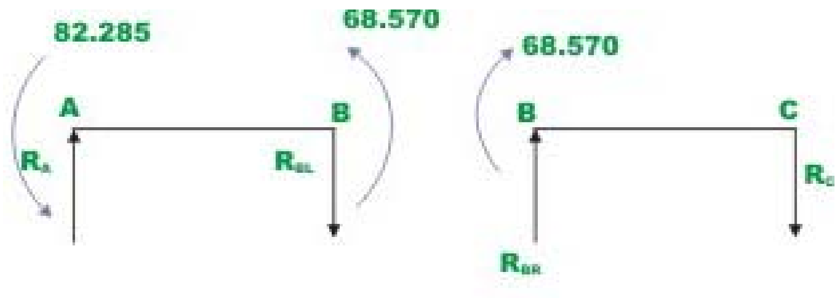


Fig 15.2d Computation of reactions

In beam AB ,

$$\sum M_B = 0, \; R_A = 30.171\,\text{kN}(\uparrow)$$

$$R_{BL} = -30.171\,\text{kN}(\downarrow)$$

$$R_{BR} = -13.714\,\text{kN}(\downarrow)$$

$$R_C = 13.714\,\text{kN}(\uparrow)$$

The shear force and bending moment diagram is shown in Fig.15.2e and elastic curve is shown in Fig.15.2f.

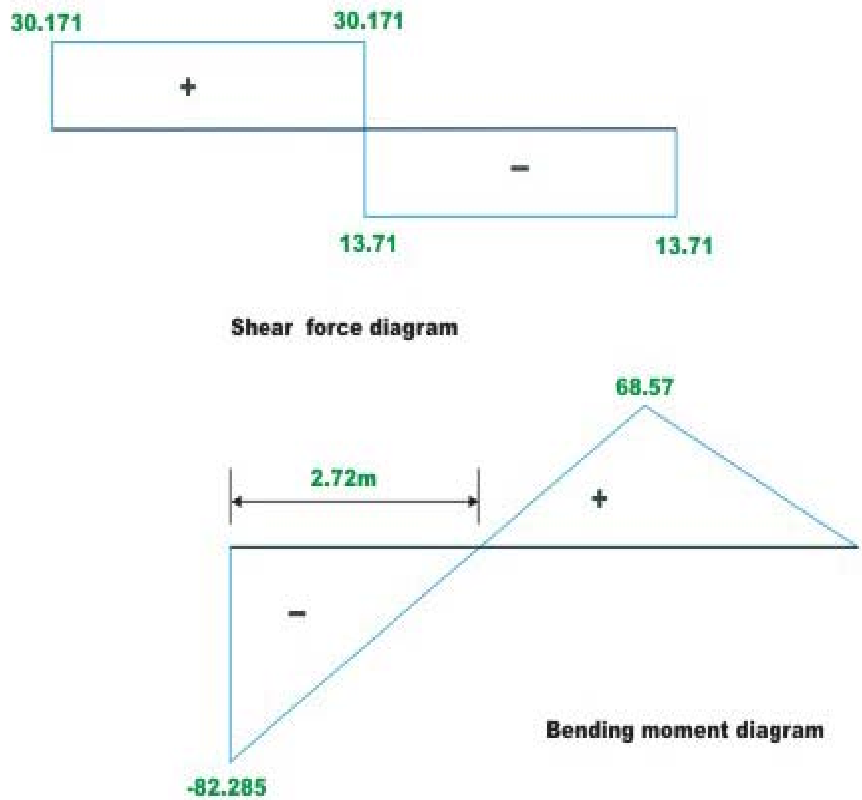
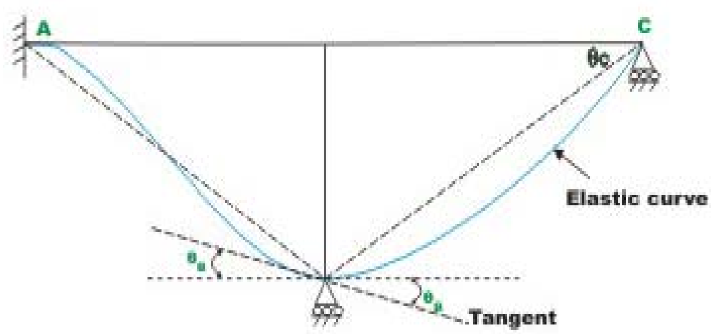


Figure 15.2e Shear force and bending moment diagram



15.2 f Elasctic curve

Example 15.2

A continuous beam *ABCD* is carrying a uniformly distributed load of 5 kN/m as shown in Fig.15.3a. Compute reactions and draw shear force and bending moment diagram due to following support settlements.

Support B 0.005m vertically downwards
Support C 0.01 m vertically downwards
Assume E =200 GPa, $I = 1.35 \times 10^{-3} m^4$

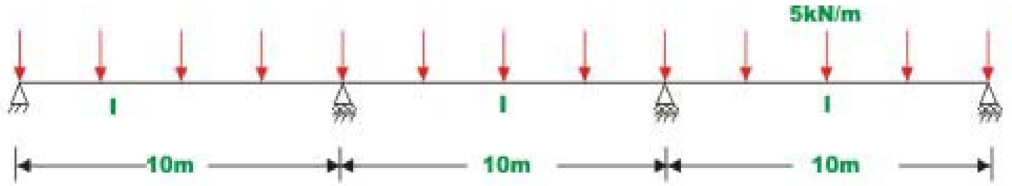


Fig 15.3a Continuous beam of Example 15.2

In the above continuous beam, four rotations $\theta_A, \theta_B, \theta_C$ and θ_D are to be evaluated. One equilibrium equation can be written at each support.Hence, solving the four equilibrium equations, the rotations are evaluated and hence the moments from slope-deflection equations. Now consider the kinematically restrained beam as shown in Fig.15.3b.

Referring to standard tables the fixed end moments may be evaluated .Otherwise one could obtain fixed end moments from force method of analysis. The fixed end moments in the present case are (vide fig.15.3b)

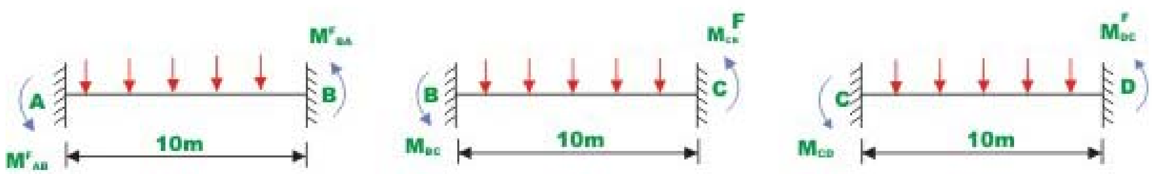


Fig 15.3b Kinematically restrained beam

$$M_{AB}^F = 41.667 \text{ kN.m}$$

$$M_{BA}^F = -41.667 \text{ kN.m} \text{ (clockwise)}$$

$$M_{BC}^F = 41.667 \text{ kN.m} \text{ (counterclockwise)}$$

$$M_{CB}^F = -41.667 \text{ kN.m} \text{ (clockwise)}$$

$$M_{CD}^F = 41.667 \text{ kN.m} \text{ (counterclockwise)}$$

$$M_{DC}^F = -41.667 \text{ kN.m} \text{ (clockwise)} \qquad (1)$$

In the next step, write slope-deflection equations for each span. In the span AB, B is below A and hence the chord joining AB' rotates in the clockwise direction (see Fig.15.3c)

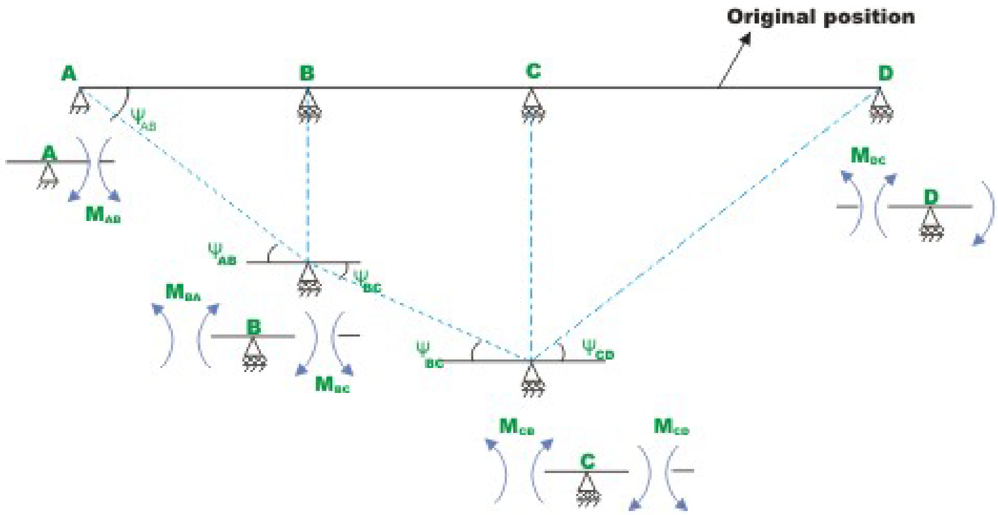


Fig 15.3c New support positions and free body diagrams of support

$\psi_{AB} = \frac{0-0.005}{10} = -0.0005$ radians (negative as the chord AB' rotates in the clockwise direction from the original direction)

$\psi_{BC} = -0.0005$ radians (negative as the chord $B'C'$ rotates in the clockwise direction)

$\psi_{CD} = \frac{0.01}{10} = 0.001$ radians (positive as the chord $C'D$ rotates in the counter clockwise direction from the original direction) (2)

Now, writing the expressions for the span end moments, for each of the spans,

$$M_{AB} = 41.667 + 0.2EI(2\theta_A + \theta_B + 0.0005)$$
$$M_{BA} = -41.667 + 0.2EI(2\theta_B + \theta_A + 0.0005)$$

$$M_{BC} = 41.667 + 0.2EI(2\theta_B + \theta_C + 0.0005)$$
$$M_{CB} = -41.667 + 0.2EI(2\theta_C + \theta_B + 0.0005)$$

$$M_{CD} = 41.667 + 0.2EI(2\theta_C + \theta_D - 0.001)$$
$$M_{DC} = -41.667 + 0.2EI(2\theta_D + \theta_C - 0.001) \qquad (3)$$

For the present problem, four joint equilibrium equations can be written, one each for each of the supports. They are

$$\sum M_A = 0 \Rightarrow M_{AB} = 0$$

$$\sum M_B = 0 \Rightarrow M_{BA} + M_{BC} = 0$$

$$\sum M_C = 0 \Rightarrow M_{CB} + M_{CD} = 0$$

$$\sum M_D = 0 \Rightarrow M_{DC} = 0 \qquad (4)$$

Substituting the values of beam end moments from equations (3) in equation (4), four equations are obtained in four unknown rotations $\theta_A, \theta_B, \theta_C$ and θ_D.They are,

$$(EI = 200 \times 10^3 \times 1.35 \times 10^{-6} = 270{,}000 \text{ kN.m}^2)$$

$$2\theta_A + \theta_B = -1.2716 \times 10^{-3}$$

$$\theta_A + 4\theta_B + \theta_C = -0.001$$

$$\theta_B + 4\theta_C + \theta_D = 0.0005$$

$$\theta_C + 2\theta_D = 1.7716 \times 10^{-3} \qquad (5)$$

Solving the above sets of simultaneous equations, values of $\theta_A, \theta_B, \theta_C$ and θ_D are evaluated.

$$\theta_A = -5.9629 \times 10^{-4} \quad \text{radians}$$

$$\theta_B = -7.9013 \times 10^{-5} \quad \text{radians}$$

$$\theta_C = -8.7653 \times 10^{-5} \quad \text{radians}$$

$$\theta_D = 9.2963 \times 10^{-4} \quad \text{radians} \qquad (6)$$

Substituting the values in slope-deflection equations the beam end moments are evaluated.

$$M_{AB} = 41.667 + 0.2 \times 270{,}000\{2(-5.9629 \times 10^{-4}) + (-7.9013 \times 10^{-5}) + 0.0005)\} = 0$$

$$M_{BA} = -41.667 + 0.2 \times 270{,}000\{2(-7.9013 \times 10^{-5}) - 5.9629 \times 10^{-4} + 0.0005\} = -55.40 \text{ kN.m}$$

$$M_{BC} = 41.667 + 0.2 \times 270{,}000\{2(-7.9013 \times 10^{-5}) + (-8.7653 \times 10^{-5}) + 0.0005\} = 55.40 \text{ kN.m}$$

$$M_{CB} = -41.667 + 0.2 \times 270{,}000\{2(-8.765 \times 10^{-5}) - 7.9013 \times 10^{-5} + 0.0005\} = -28.40 \text{ kN.m}$$

$$M_{CD} = 41.667 + 0.2 \times 270{,}000\{2 \times (-8.765 \times 10^{-5}) + 9.2963 \times 10^{-4} - 0.001\} = 28.40 \text{ kN.m}$$

$$M_{DC} = -41.667 + 0.2 \times 270{,}000\{2 \times 9.2963 \times 10^{-4} - 8.7653 \times 10^{-5} - 0.001\} = 0 \text{ kN.m} \quad (7)$$

Reactions are obtained from equilibrium equations. Now consider the free body diagram of the beam with end moments and external loads as shown in Fig.15.3d.

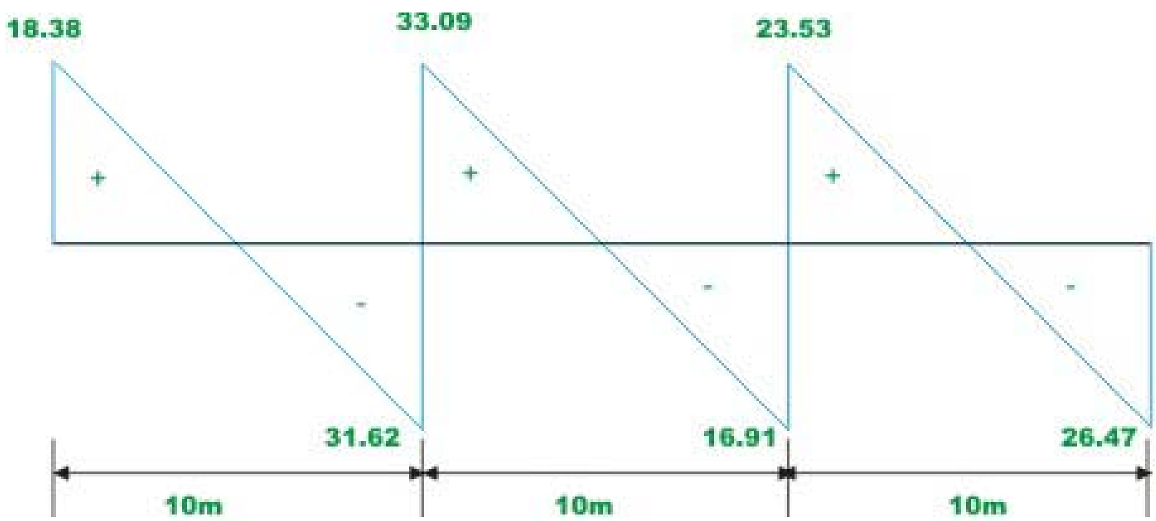


Fig 15.3d Shear force diagram

$$R_A = 19.46 \text{ kN}(\uparrow)$$

$$R_{BL} = 30.54 \text{ kN}(\uparrow)$$

$$R_{BR} = 27.7 \text{ kN}(\uparrow)$$

$$R_{CL} = 22.3 \text{ kN}(\uparrow)$$

$$R_{CR} = 27.84 \text{ kN}(\uparrow)$$

$$R_{D} = 22.16 \text{ kN}(\uparrow)$$

The shear force and bending moment diagrams are shown in Fig.15.5e.

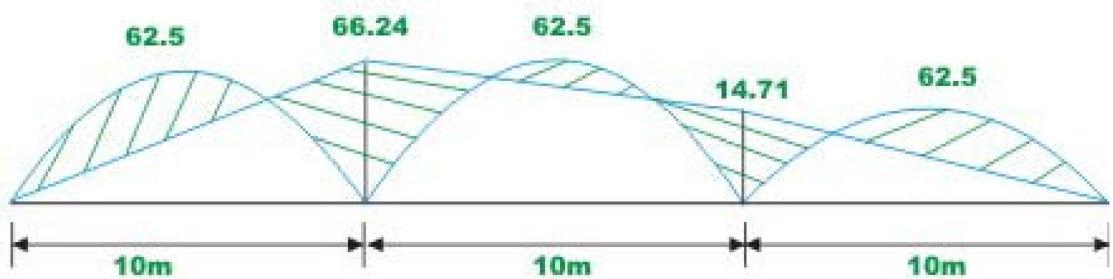


Fig. 15.3e Bending moment diagram

Summary

In this lesson, slope-deflection equations are derived for the case of beam with yielding supports. Moments developed at the ends are related to rotations and support settlements. The equilibrium equations are written at each support. The continuous beam is solved using slope-deflection equations. The deflected shape of the beam is sketched. The bending moment and shear force diagrams are drawn for the examples solved in this lesson.