Module 2

Analysis of Statically Indeterminate Structures by the Matrix Force Method

Lesson
8
The Force Method of Analysis: Beams

Instructional Objectives

After reading this chapter the student will be able to

1. Solve statically indeterminate beams of degree more than one.
2. To solve the problem in matrix notation.
3. To compute reactions at all the supports.
4. To compute internal resisting bending moment at any section of the continuous beam.

8.1 Introduction

In the last lesson, a general introduction to the force method of analysis is given. Only, beams, which are statically indeterminate to first degree, were considered. If the structure is statically indeterminate to a degree more than one, then the approach presented in the previous example needs to be organized properly. In the present lesson, a general procedure for analyzing statically indeterminate beams is discussed.

8.2 Formalization of Procedure

Towards this end, consider a two-span continuous beam as shown in Fig. 8.1a. The flexural rigidity of this continuous beam is assumed to be constant and is taken as EI. Since, the beam is statically indeterminate to second degree, it is required to identify two redundant reaction components, which need be released to make the beam statically determinate.

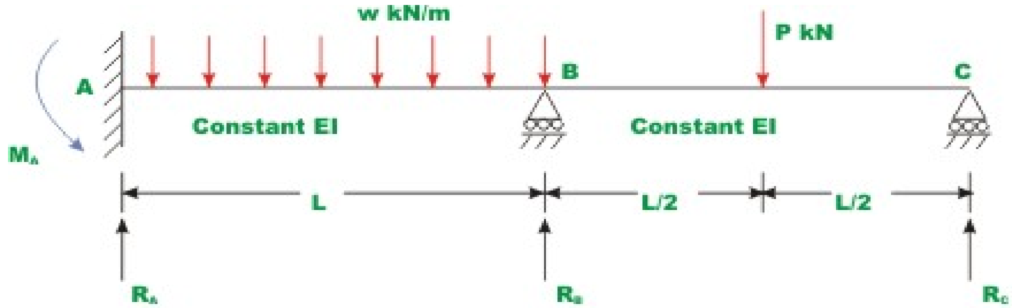

Fig. 8.1(a). Continuous beam.

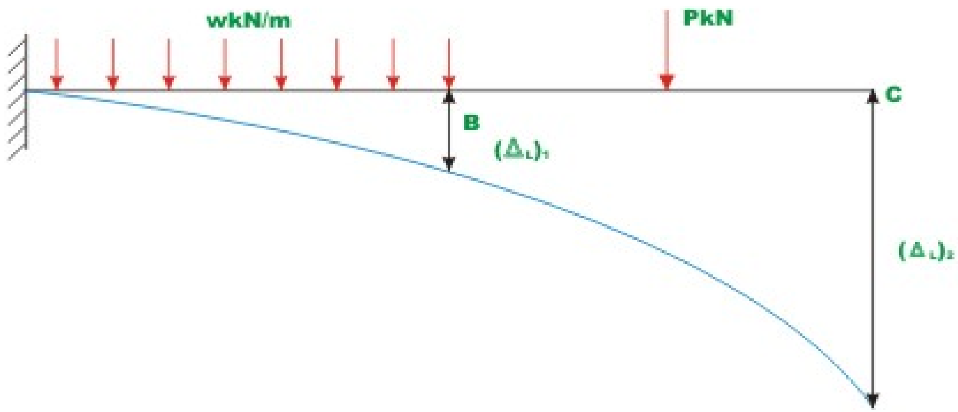

Fig. 8.1(b). Primary structure with applied loading.

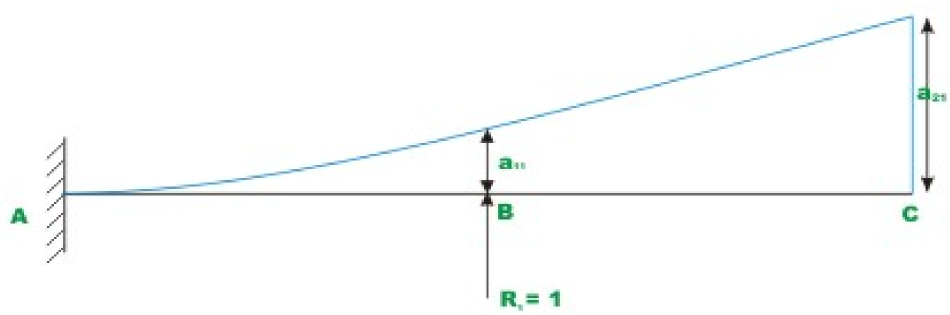

Fig. 8.1(c) Primary structure with unit load applied along R_1

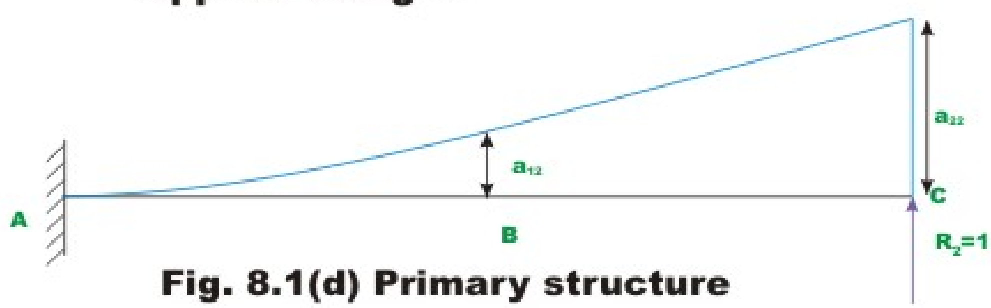

Fig. 8.1(d) Primary structure with unit load applied along R_2

The redundant reactions at A and B are denoted by R_1 and R_2 respectively. The released structure (statically determinate structure) with applied loading is shown in Fig. 8.1b. The deflection of primary structure at B and C due to applied loading is denoted by $(\Delta_L)_1$ and $(\Delta_L)_2$ respectively. Throughout this module $(\Delta_L)_i$ notation is used to denote deflection at i^{th} redundant due to applied loads on the determinate structure.

$$(\Delta_L)_1 = -\frac{wL^4}{8EI} - \frac{7PL^3}{12EI} \tag{8.1a}$$

$$(\Delta_L)_2 = -\frac{7wL^4}{24EI} - \frac{27PL^3}{16EI} \tag{8.1b}$$

In fact, the subscript 1 and 2 represent, locations of redundant reactions released. In the present case $R_A(=R_1)$ and $R_B(=R_2)$ respectively. In the present and subsequent lessons of this module, the deflections and the reactions are taken to be positive in the upward direction. However, it should be kept in mind that the positive sense of the redundant can be chosen arbitrarily. The deflection of the point of application of the redundant should likewise be considered positive when acting in the same sense.

For writing compatibility equations at B and C, it is required to know deflection of the released structure at B and C due to external loading and due to redundants. The deflection at B and C due to external loading can be computed easily. Since redundants R_1 and R_2 are not known, in the first step apply a unit load in the direction of R_1 and compute deflection, a_{11} at B, and deflection, a_{21} at C, as shown in Fig.8.1c. Now deflections at B and C of the given released structure due to redundant R_1 are,

$$(\Delta_R)_{11} = a_{11}\ R_1 \tag{8.2a}$$

$$(\Delta_R)_{21} = a_{21}\ R_1 \tag{8.2b}$$

In the second step, apply unit load in the direction of redundant R_2 and compute deflection at B (point 1), a_{12} and deflection at C, a_{22} as shown in Fig 8.1d. It may be recalled that the flexibility coefficient a_{ij} is the deflection at i due to unit value of force applied at j. Now deflections of the primary structure (released structure) at B and C due to redundant R_2 is

$$(\Delta_R)_{12} = a_{12}\ R_2 \tag{8.3a}$$

$$(\Delta_R)_{22} = a_{22}\ R_2 \tag{8.3b}$$

It is observed that, in the actual structure, the deflections at joints B and C is zero. Now the total deflections at B and C of the primary structure due to applied external loading and redundants R_1 and R_2 is,

$$\Delta_1 = (\Delta_L)_1 + a_{11}R_1 + a_{12}R_2 \tag{8.4a}$$

$$\Delta_2 = (\Delta_L)_2 + a_{21}R_1 + a_{22}R_2 \tag{8.4b}$$

The equation (8.4a) represents the total displacement at B and is obtained by superposition of three terms:

1) Deflection at B due to actual load acting on the statically determinate structure,
2) Displacement at B due to the redundant reaction R_1 acting in the positive direction at B (point 1) and
3) Displacement at B due to the redundant reaction R_2 acting in the positive direction at C.

The second equation (8.4b) similarly represents the total deflection at C. From the physics of the problem, the compatibility condition can be written as,

$$\Delta_1 = (\Delta_L)_1 + a_{11}R_1 + a_{12}R_2 = 0 \tag{8.5a}$$

$$\Delta_2 = (\Delta_L)_2 + a_{21}R_1 + a_{22}R_2 = 0 \tag{8.5b}$$

The equation (8.5a) and (8.5b) may be written in matrix notation as follows,

$$\begin{Bmatrix} (\Delta_L)_1 \\ (\Delta_L)_2 \end{Bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{8.6a}$$

$$\{(\Delta_L)_1\} + [A]\{R\} = \{0\} \tag{8.6b}$$

In which,

$$\{(\Delta_L)_1\} = \begin{Bmatrix} (\Delta_L)_1 \\ (\Delta_L)_2 \end{Bmatrix}; \ [A] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ and } \{R\} = \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix}$$

Solving the above set of algebraic equations, one could obtain the values of redundants, R_1 and R_2.

$$\{R\} = -[A]^{-1}\{\Delta_L\} \tag{8.7}$$

In the above equation the vectors $\{\Delta_L\}$ contains the displacement values of the primary structure at point 1 and 2, $[A]$ is the flexibility matrix and $\{R\}$ is column vector of redundants required to be evaluated. In equation (8.7) the inverse of the flexibility matrix is denoted by $[A]^{-1}$. In the above example, the structure is indeterminate to second degree and the size of flexibility matrix is 2×2. In general, if the structure is redundant to a degree n, then the flexibility matrix is of the order $n \times n$. To demonstrate the procedure to evaluate deflection, consider the problem given in Fig. 8.1a, with loading as given below

$$w = w \,; \qquad\qquad P = wL \tag{8.8a}$$

Now, the deflection $(\Delta_L)_1$ and $(\Delta_L)_2$ of the released structure can be evaluated from the equations (8.1a) and (8.1b) respectively. Then,

$$(\Delta_L)_1 = -\frac{wL^4}{8EI} - \frac{7wL^4}{12EI} = -\frac{17wL^4}{24EI} \tag{8.8b}$$

$$(\Delta_L)_2 = -\frac{7wL^4}{24EI} - \frac{27wL^4}{16EI} = -\frac{95wL^4}{48EI} \tag{8.8c}$$

The negative sign indicates that both deflections are downwards. Hence the vector $\{\Delta_L\}$ is given by

$$\{\Delta_L\} = -\frac{wL^4}{48EI}\begin{Bmatrix} 34 \\ 95 \end{Bmatrix} \tag{8.8d}$$

The flexibility matrix is determined from referring to figures 8.1c and 8.1d. Thus, when the unit load corresponding to R_1 is acting at B, the deflections are,

$$a_{11} = \frac{L^3}{3EI}\,, \qquad\qquad a_{21} = \frac{5L^3}{6EI} \tag{8.8e}$$

Similarly when the unit load is acting at C,

$$a_{12} = \frac{5L^3}{6EI}\,, \qquad\qquad a_{22} = \frac{8L^3}{3EI} \tag{8.8f}$$

The flexibility matrix can be written as,

$$[A] = \frac{L^3}{6EI}\begin{bmatrix} 2 & 5 \\ 5 & 16 \end{bmatrix} \tag{8.8g}$$

The inverse of the flexibility matrix can be evaluated by any of the standard method. Thus,

$$[A]^{-1} = \frac{6EI}{7L^3}\begin{bmatrix} 16 & -5 \\ -5 & 2 \end{bmatrix} \tag{8.8h}$$

Now using equation (8.7) the redundants are evaluated. Thus,

$$\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \frac{6EI}{7L^3} \times \frac{wL^4}{48EI}\begin{bmatrix} 16 & -5 \\ -5 & 2 \end{bmatrix}\begin{Bmatrix} 34 \\ 95 \end{Bmatrix}$$

Hence, $R_1 = \frac{69}{56}wL$ and $R_2 = \frac{20}{56}wL$ (8.8i)

Once the redundants are evaluated, the other reaction components can be evaluated by static equations of equilibrium.

Example 8.1

Calculate the support reactions in the continuous beam ABC due to loading as shown in Fig. 8.2a. Assume EI to be constant throughout.

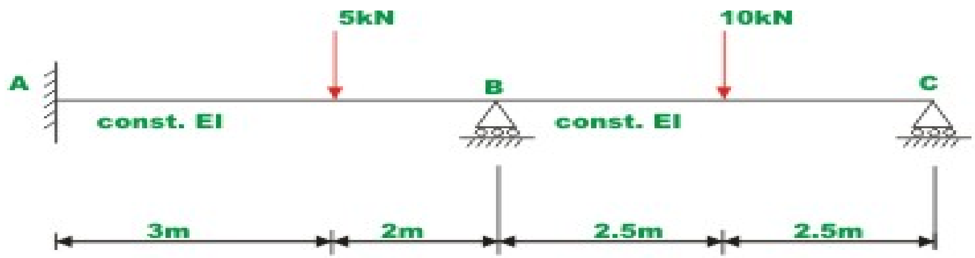

Fig. 8.2 (a) Example 8.2

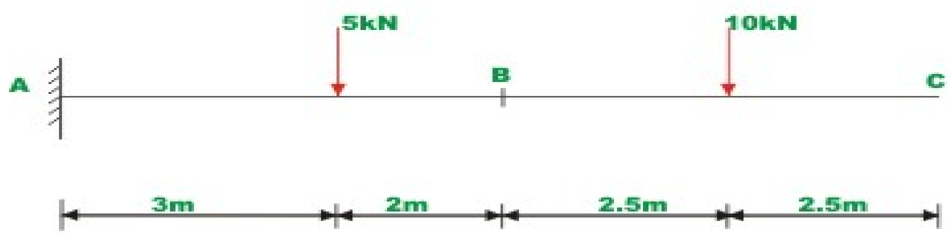

Fig. 8.2 (b) Primary structure with external load

Select two reactions viz, at $B(R_1)$ and $C(R_2)$ as redundants, since the given beam is statically indeterminate to second degree. In this case the primary structure is a cantilever beam AC. The primary structure with a given loading is shown in Fig. 8.2b.

In the present case, the deflections $(\Delta_L)_1$, and $(\Delta_L)_2$ of the released structure at B and C can be readily calculated by moment-area method. Thus,

$$(\Delta_L)_1 = -\frac{819.16}{EI}$$

and
$$(\Delta_L)_2 = -\frac{2311.875}{EI} \tag{1}$$

For the present problem the flexibility matrix is,

$$a_{11} = \frac{125}{3EI} \qquad a_{21} = \frac{625}{6EI}$$

$$a_{12} = \frac{625}{6EI} \qquad a_{22} = \frac{1000}{3EI} \tag{2}$$

In the actual problem the displacements at B and C are zero. Thus the compatibility conditions for the problem may be written as,

$$\begin{aligned} a_{11}R_1 + a_{12}R_2 + (\Delta_L)_1 &= 0 \\ a_{21}R_1 + a_{22}R_2 + (\Delta_L)_2 &= 0 \end{aligned} \tag{3}$$

$$\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \frac{3EI}{27343.75} \begin{bmatrix} 1000 & -312.5 \\ -312.5 & 125 \end{bmatrix} \times \frac{1}{EI} \begin{Bmatrix} 819.16 \\ 2311.875 \end{Bmatrix} \tag{5}$$

Substituting the value of E and I in the above equation,

$$R_1 = 10.609\text{kN} \quad \text{and} \quad R_2 = 3.620 \ \text{kN}$$

Using equations of static equilibrium,

$$R_3 = 0.771 \ \text{kN} \qquad \text{and} \quad R_4 = -0.755 \ \text{kN.m}$$

Example 8.2

A clamped beam AB of constant flexural rigidity is shown in Fig. 8.3a. The beam is subjected to a uniform distributed load of w kN/m and a central concentrated moment $M = wL^2$ kN.m. Draw shear force and bending moment diagrams by force method.

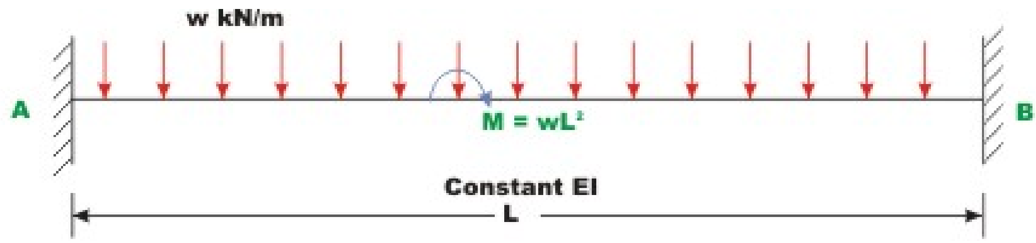

Fig. 8.3(a) Clamped beam (Example 8.1)

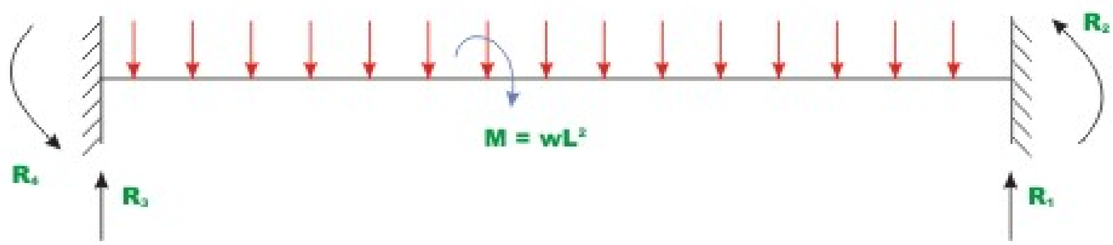

Fig. 8.3(b) Clamped beam with R₁ and R₂ as redundants

Select vertical reaction (R_1) and the support moment (R_2) at B as the redundants. The primary structure in this case is a cantilever beam which could be obtained by releasing the redundants R_1 and R_2. The R_1 is assumed to be positive in the upward direction and R_2 is assumed to be positive in the counterclockwise direction. Now, calculate deflection at B due to only applied loading. Let $(\Delta_L)_1$ be the transverse deflection at B and $(\Delta_L)_2$ be the slope at B due to external loading. The positive directions of the selected redundants are shown in Fig. 8.3b.

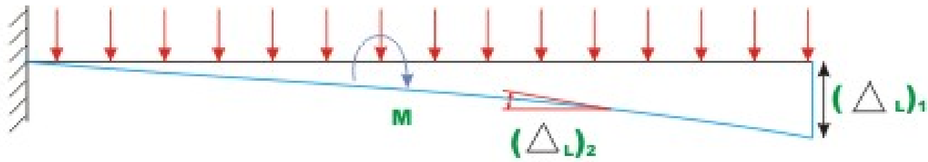

Fig. 8.3(c) Primary structure with external loading

Fig. 8.3(d)Primary structure with unit load along R_1

Fig. 8.3 (e) Primary structure with unit moment along R_2

Fig.8.3f Reaction

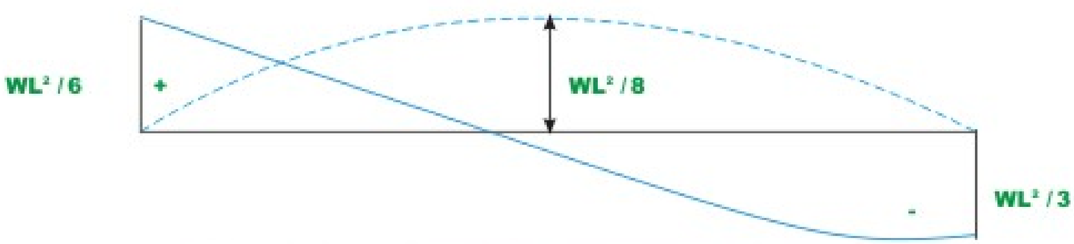

Fig 8.3(g) Bending moment diagram

Fig.8.3(h) Shear force diagram.

The deflection $(\Delta_L)_1$ and $(\Delta_L)_2$ of the released structure can be evaluated from unit load method. Thus,

$$(\Delta_L)_1 = -\frac{wL^4}{8EI} - \frac{3wL^4}{8EI} = -\frac{wL^4}{2EI} \tag{1}$$

and $$(\Delta_L)_2 = -\frac{wL^3}{6EI} - \frac{wL^3}{2EI} = -\frac{2wL^3}{3EI} \tag{2}$$

The negative sign indicates that $(\Delta_L)_1$ is downwards and rotation $(\Delta_L)_2$ is clockwise. Hence the vector $\{\Delta_L\}$ is given by

$$\{\Delta_L\} = -\frac{wL^3}{6EI}\begin{Bmatrix} 3L \\ 4 \end{Bmatrix} \tag{3}$$

The flexibility matrix is evaluated by first applying unit load along redundant R_1 and determining the deflections a_{11} and a_{21} corresponding to redundants R_1 and R_2 respectively (see Fig. 8.3d). Thus,

$$a_{11} = \frac{L^3}{3EI} \qquad \text{and } a_{21} = \frac{L^2}{2EI} \tag{4}$$

Similarly, applying unit load in the direction of redundant R_2, one could evaluate flexibility coefficients a_{12} and a_{22} as shown in Fig. 8.3c.

$$a_{12} = \frac{L^2}{2EI} \qquad \text{and } a_{22} = \frac{L}{EI} \tag{5}$$

Now the flexibility matrix is formulated as,

$$[A] = \frac{L}{6EI}\begin{bmatrix} 2L^2 & 3L \\ 3L & 6 \end{bmatrix} \tag{6}$$

The inverse of flexibility matrix is formulated as,

$$[A]^{-1} = \frac{6EI}{3L^3}\begin{bmatrix} 6 & -3L \\ -3L & 2L^2 \end{bmatrix}$$

The redundants are evaluated from equation (8.7). Hence,

$$\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = -\frac{6EI}{3L^3}\begin{bmatrix} 6 & -3L \\ -3L & 2L^2 \end{bmatrix} \times \left(-\frac{wL^3}{6EI}\right)\begin{Bmatrix} 3L \\ 4 \end{Bmatrix}$$

$$= \frac{w}{3}\begin{Bmatrix} 6L \\ -L^2 \end{Bmatrix}$$

$$R_1 = 2wL \qquad and \quad R_2 = -\frac{wL^2}{3} \tag{7}$$

The other two reactions (R_3 and R_4) can be evaluated by equations of statics. Thus,

$$R_4 = M_A = -\frac{wL^2}{6} \qquad \text{and} \quad R_1 = R_A = -wL \tag{8}$$

The bending moment and shear force diagrams are shown in Fig. 8.3g and Fig.8.3h respectively.

Summary

In this lesson, statically indeterminate beams of degree more than one is solved systematically using flexibility matrix method. Towards this end matrix notation is adopted. Few illustrative examples are solved to illustrate the procedure. After analyzing the continuous beam, reactions are calculated and bending moment diagrams are drawn.