

# Module 2

## Analysis of Statically Indeterminate Structures by the Matrix Force Method

Lesson

11

The Force Method of  
Analysis: Frames

## Instructional Objectives

After reading this chapter the student will be able to

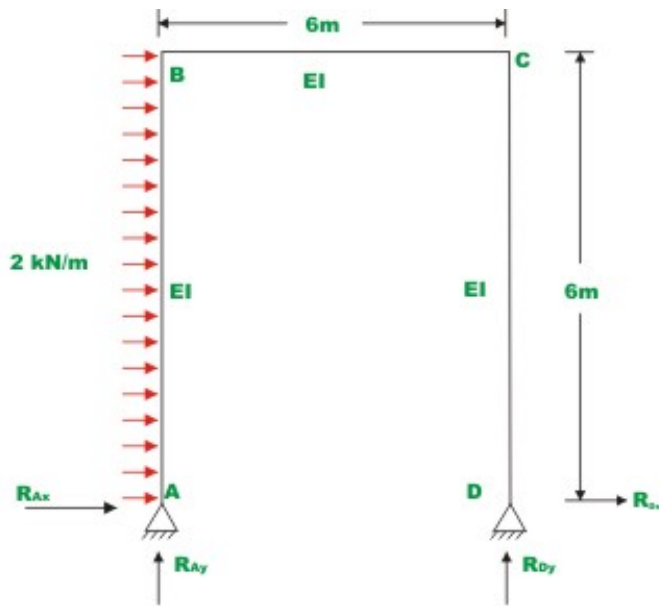
1. Analyse the statically indeterminate plane frame by force method.
2. Analyse the statically indeterminate plane frames undergoing support settlements.
3. Calculate the static deflections of a primary structure (released frame) under external loads.
4. Write compatibility equations of displacements for the plane deformations.
5. Compute reaction components of the indeterminate frame.
6. Draw shear force and bending moment diagrams for the frame.
7. Draw qualitative elastic curve of the frame.

### 11.1 Introduction

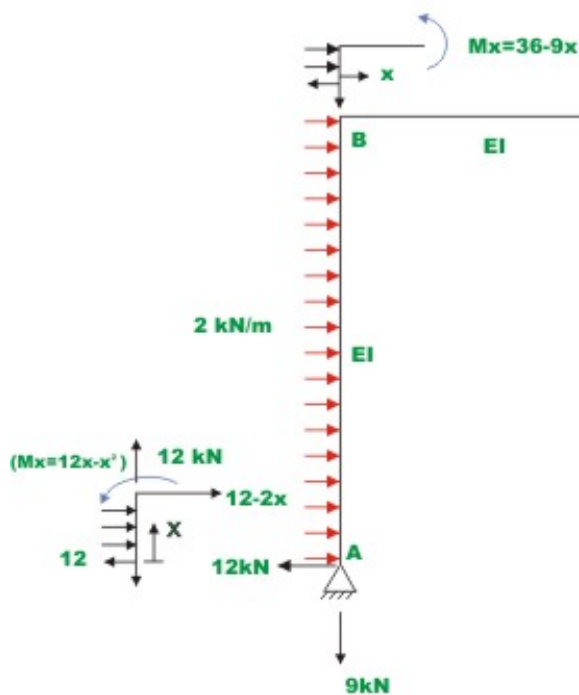
The force method of analysis can readily be employed to analyze the indeterminate frames. The basic steps in the analysis of indeterminate frame by force method are the same as that discussed in the analysis of indeterminate beams in the previous lessons. Under the action of external loads, the frames undergo axial and bending deformations. Since the axial rigidity of the members is much higher than the bending rigidity, the axial deformations are much smaller than the bending deformations and are normally not considered in the analysis. The compatibility equations for the frame are written with respect to bending deformations only. The following examples illustrate the force method of analysis as applied to indeterminate frames.

#### Example 11.1

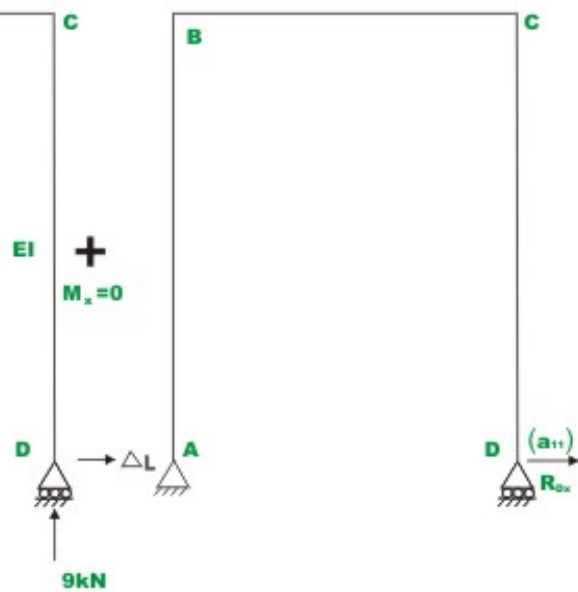
Analyse the rigid frame shown in Fig.11.1a and draw the bending moment diagram. Young's modulus  $E$  and moment of inertia  $I$  are constant for the plane frame. Neglect axial deformations.



**Fig 11.1 (a) Example 11.1**



**Fig 11.1 (b) Primary Structure**



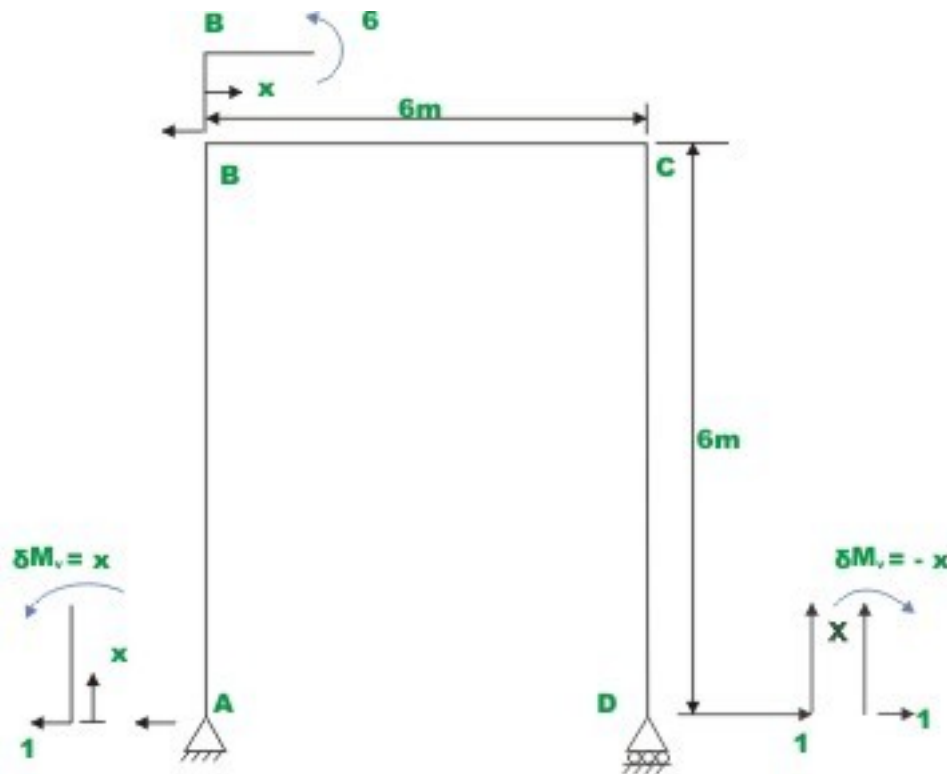
**Fig 11.1 (c) Primary Structure with redundant  $R_{dx}$**

The given one- storey frame is statically indeterminate to degree one. In the present case, the primary structure is one that is hinged at  $A$  and roller supported at  $D$ . Treat horizontal reaction at  $D$ ,  $R_{Dx}$  as the redundant. The primary structure (which is stable and determinate) is shown in Fig.11.1.b. The compatibility condition of the problem is that the horizontal deformation of the primary structure (Fig.11.1.b) due to external loads plus the horizontal deformation of the support  $D$ , due to redundant

$R_{Dx}$  (vide Fig.11.1.b) must vanish. Calculate deformation  $a_{11}$  due to unit load at  $D$  in the direction of  $R_{Dx}$ . Multiplying this deformation  $a_{11}$  with  $R_{Dx}$ , the deformation due to redundant reaction can be obtained.

$$\Delta = a_{11}R_{Dx} \quad (1)$$

Now compute the horizontal deflection  $\Delta_L$  at  $D$  due to externally applied load. This can be readily determined by unit load method. Apply a unit load along  $R_{Dx}$  as shown in Fig.10.1d.



**Fig 11.1 (d) Primary Structure with unit load along  $R_{dx}$**

The horizontal deflection  $\Delta_L$  at  $D$  in the primary structure due to external loading is given by

$$\Delta_L = \int_A^D \frac{\delta M_v M}{EI} dx \quad (2)$$

where  $\delta M_v$  is the internal virtual moment resultant in the frame due to virtual load applied at  $D$  along the resultant  $R_{Dx}$  and  $M$  is the internal bending moment of the frame due to external loading (for details refer to Module 1, Lesson 5). Thus,

$$\Delta_L = \int_0^6 \frac{(12x - x^2)x}{EI} dx + \int_0^6 \frac{(36 - 9x)6}{EI} dx + \int_0^6 \frac{0(x)}{EI} dx$$

(span AB, origin at A)      (span BC, origin at B)      (span DC, origin at D)

$$\Delta_L = \frac{864}{EI} \tag{3}$$

In the next step, calculate the displacement  $a_{11}$  at  $D$  when a real unit load is applied at  $D$  in the direction of  $R_{Dx}$  (refer to Fig.11.1 d). Please note that the same Fig. 11.1d is used to represent unit virtual load applied at  $D$  and real unit load applied at  $D$ . Thus,

$$a_{11} = \int_A^D \frac{\delta m_v m}{EI} dx$$

$$= \int_0^6 \frac{x^2 dx}{EI} + \int_0^6 \frac{36 dx}{EI} + \int_0^6 \frac{x^2 dx}{EI}$$

$$= \frac{360}{EI} \tag{4}$$

Now, the compatibility condition of the problem may be written as

$$\Delta_L + a_{11} R_{Dx} = 0 \tag{5}$$

Solving,

$$R_{Dx} = -2.40 \text{ kN} \tag{6}$$

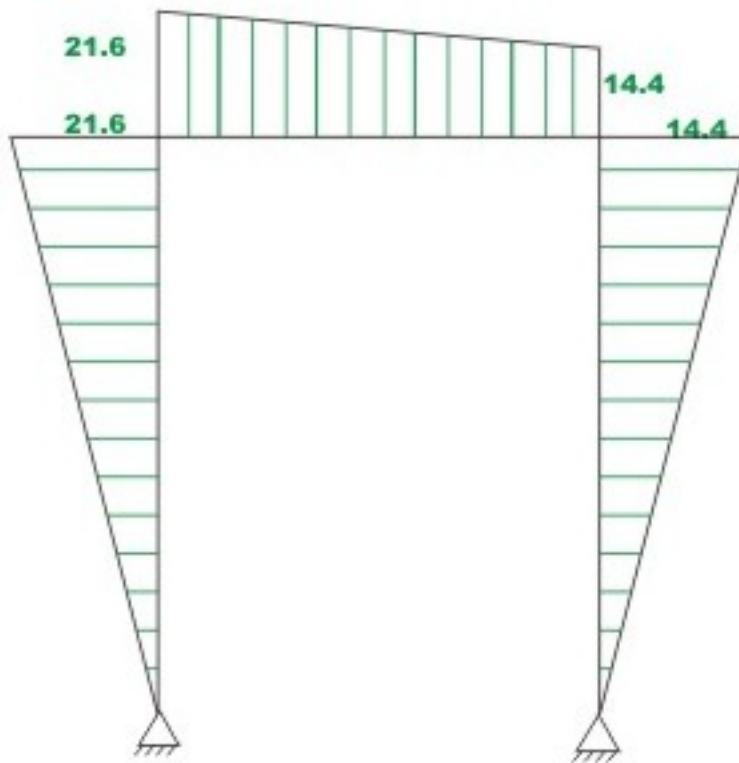
The minus sign indicates that the redundant reaction  $R_{Dx}$  acts towards left. Remaining reactions are calculated from equations of static equilibrium.

$$\sum F_x = 0 \Rightarrow R_{Ax} = -12 + 2.40 = -9.60 \text{ kN (towards left)}$$

$$\sum M_D = 0 \Rightarrow R_{Ay} = -9 \text{ kN (downwards)}$$

$$R_{Dy} = +9 \text{ kN (upwards)}$$

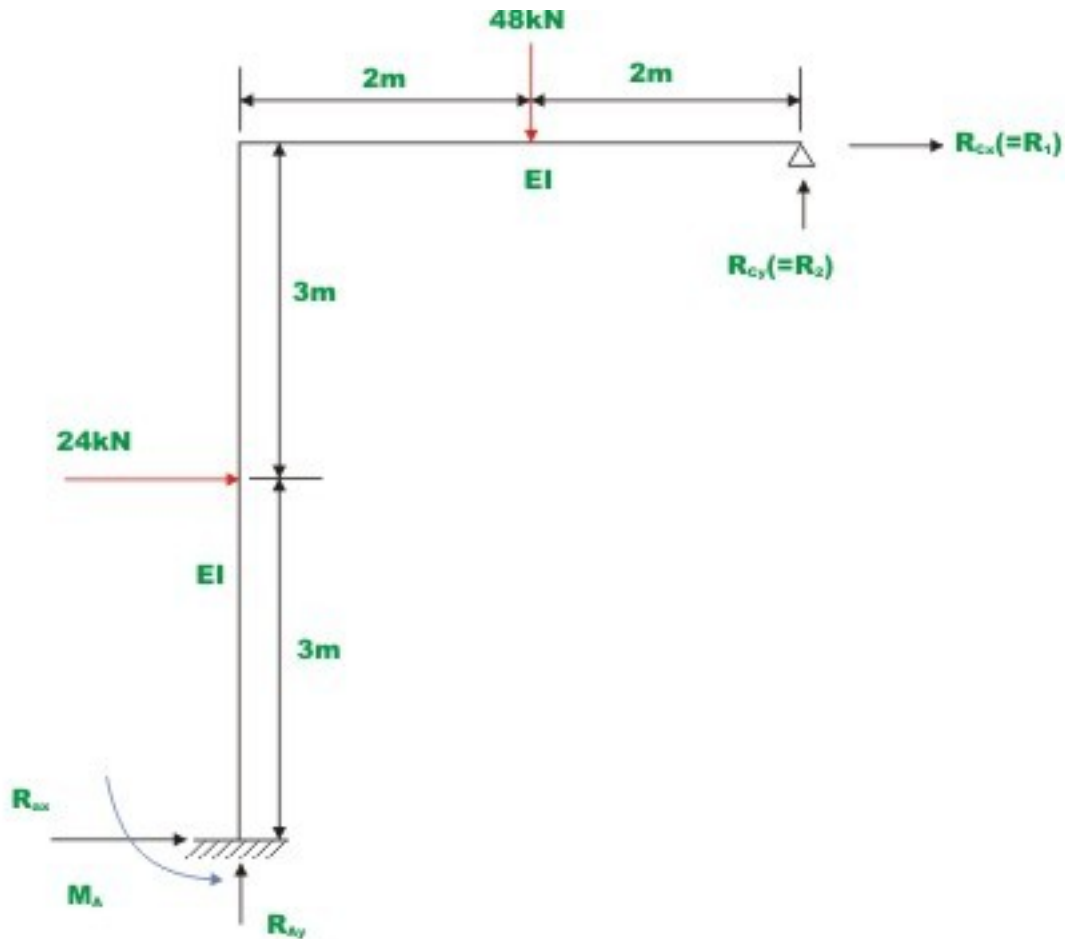
The bending moment diagram for the frame is shown in Fig. 11.1e



**Fig 11.1 (e) Bending moment diagram**

### Example 11.2

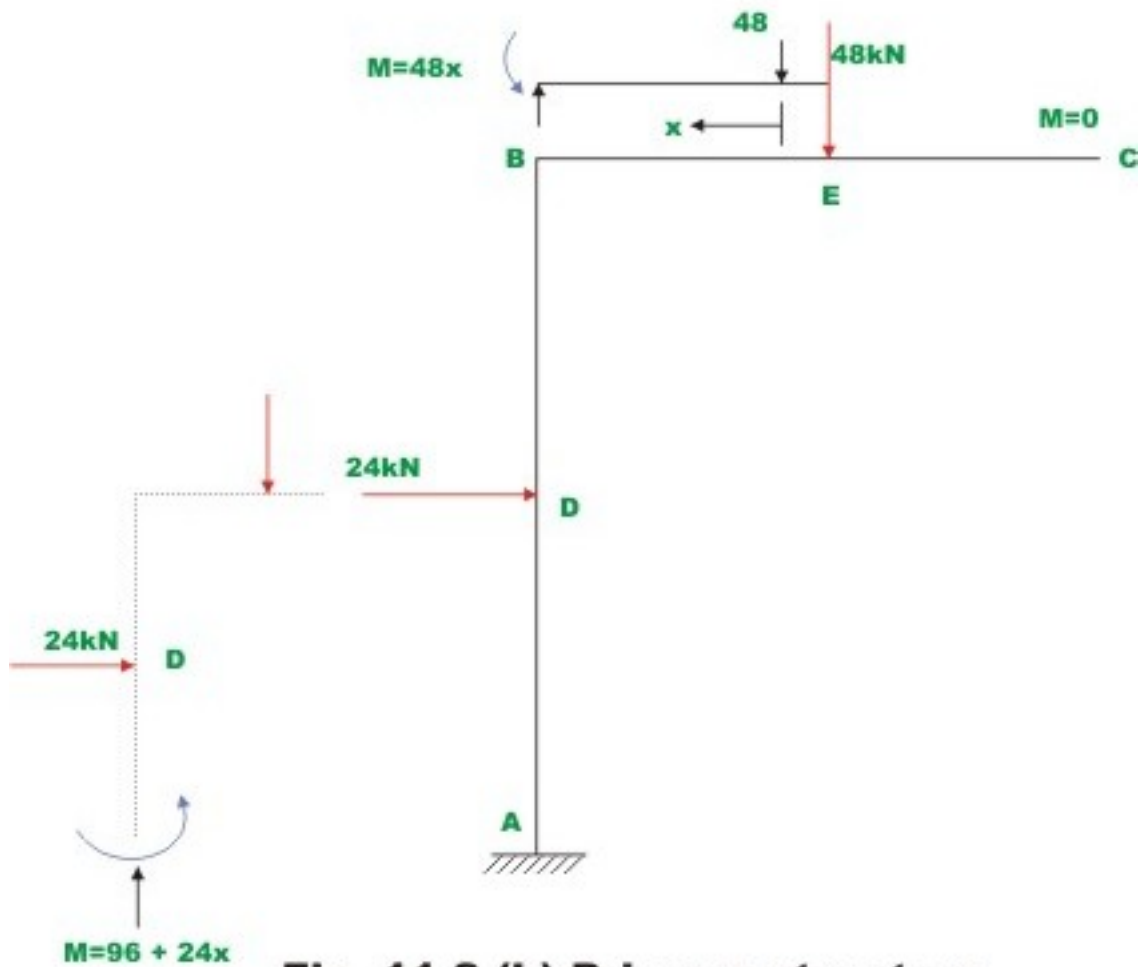
Analyze the rigid frame shown in Fig.11.2a and draw the bending moment and shear force diagram. The flexural rigidity for all members is the same. Neglect axial deformations.



**Fig 11.2 (a) Example 11.2**

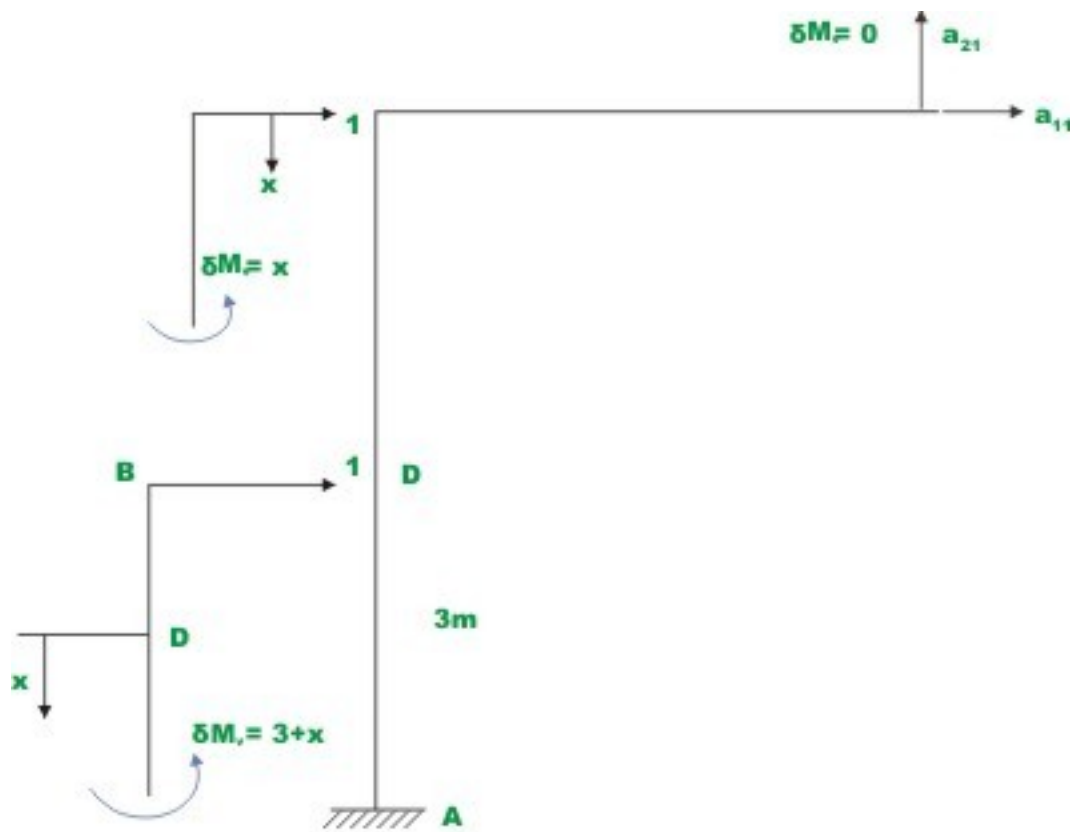
Five reactions components need to be evaluated in this rigid frame; hence it is indeterminate to second degree. Select  $R_{cx}(=R_1)$  and  $R_{cy}(=R_2)$  as the redundant reactions. Hence, the primary structure is one in which support A is fixed and the support C is free as shown in Fig.11.2b. Also, equations for moments in various spans of the frame are also given in the figure.



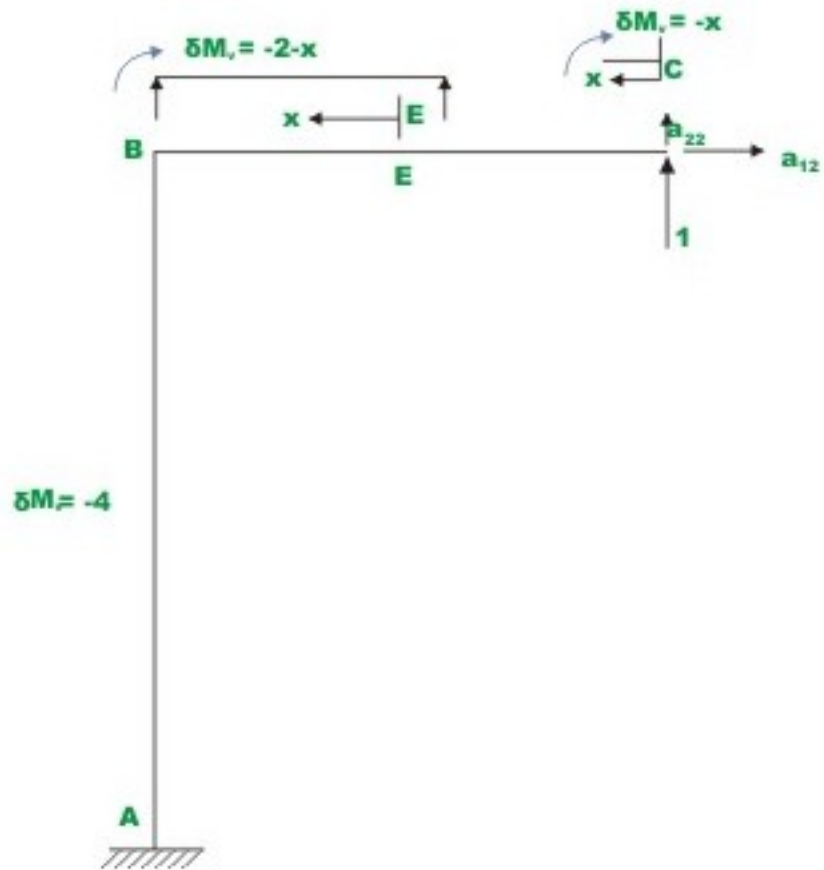


**Fig 11.2 (b) Primary structure with external load**

Calculate horizontal  $(\Delta_L)_1$  and vertical  $(\Delta_L)_2$  deflections at C in the primary structure due to external loading. This can be done by unit load method. Thus,



**Fig 11.2( c ) Primary structure with unit load along  $R_1$**



**Fig 11.2 (d) Primary structure with unit load along  $R_2$**

$$\begin{aligned}
 (\Delta_L)_1 &= \int_0^3 \frac{(96 + 24x)(3 + x)}{EI} dx + \int_0^3 \frac{96x}{EI} dx + 0 \\
 &\text{(Span DA, origin at D)} \quad \text{(Span BD, origin at B)} \quad \text{(span BC, Origin B)} \\
 &= \frac{2268}{EI} \quad (1)
 \end{aligned}$$

$$\begin{aligned}
 (\Delta_L)_2 &= \int_0^3 \frac{(96 + 24x)(-4) dx}{EI} + \int_0^3 \frac{96(-4) dx}{EI} + \int_0^2 \frac{48x(-2 - x) dx}{EI} + 0 \\
 &\text{(Span DA, origin at D)} \quad \text{(Span BD, origin at B)} \quad \text{(Span BE, origin at E)} \\
 &\text{(Span EC, Origin C)}
 \end{aligned}$$

$$(\Delta_L)_2 = -\frac{3056}{EI} \quad (2)$$

In the next step, evaluate flexibility coefficients, this is done by applying a unit load along,  $R_1$  and determining deflections  $a_{11}$  and  $a_{21}$  corresponding to  $R_1$  and  $R_2$  respectively (vide, Fig .11.2 c). Again apply unit load along  $R_2$  and evaluate deflections  $a_{22}$  and  $a_{12}$  corresponding to  $R_2$  and  $R_1$  and respectively (ref. Fig.11.2d).

$$a_{11} = \int_0^6 \frac{x^2}{EI} dx = \frac{72}{EI} \quad (3)$$

$$\begin{aligned} a_{12} = a_{21} &= \int_0^6 \frac{x(-4)}{EI} dx + 0 \\ &= \frac{72}{EI} \end{aligned} \quad (4)$$

and

$$\begin{aligned} a_{22} &= \int_0^6 \frac{16}{EI} dx + \int_0^4 \frac{(x)^2}{EI} dx \\ &= \frac{117.33}{EI} \end{aligned} \quad (5)$$

In the actual structure at C, the horizontal and vertical displacements are zero .Hence, the compatibility condition may be written as,

$$(\Delta_L)_1 = a_{11}R_1 + a_{12}R_2 = 0$$

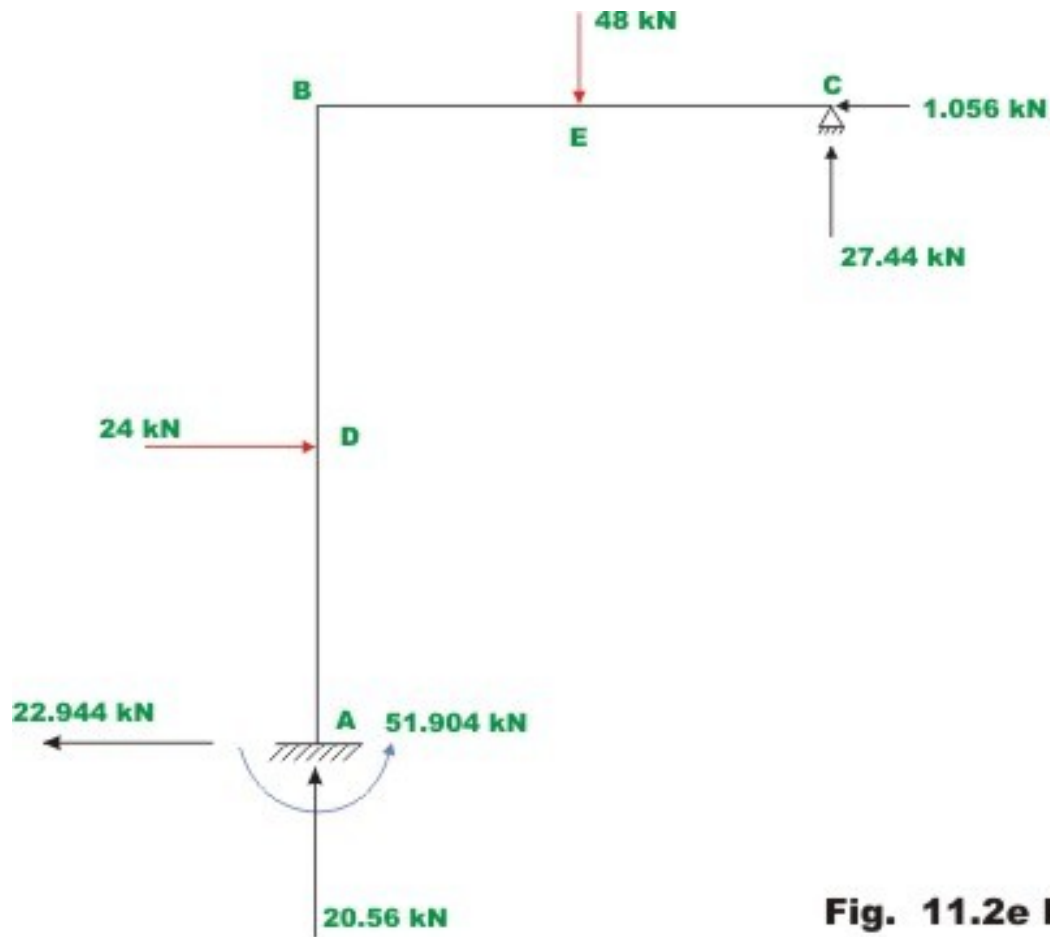
$$(\Delta_L)_2 = a_{12}R_1 + a_{22}R_2 = 0 \quad (6)$$

Substituting the values of  $(\Delta_L)_1, (\Delta_L)_2, a_{11}, a_{12}$  and  $a_{22}$  in the above equations and solving for  $R_1, R_2$  we get

$$R_1 = -1.056 \text{ kN (towards left)}$$

$$R_2 = 27.44 \text{ kN (upwards)}$$

The remaining reactions are calculated from equations of statics and they are shown in Fig 11.2e. The bending moment and shear force diagrams are shown in Fig. 11.2f.



**Fig. 11.2e Reaction**

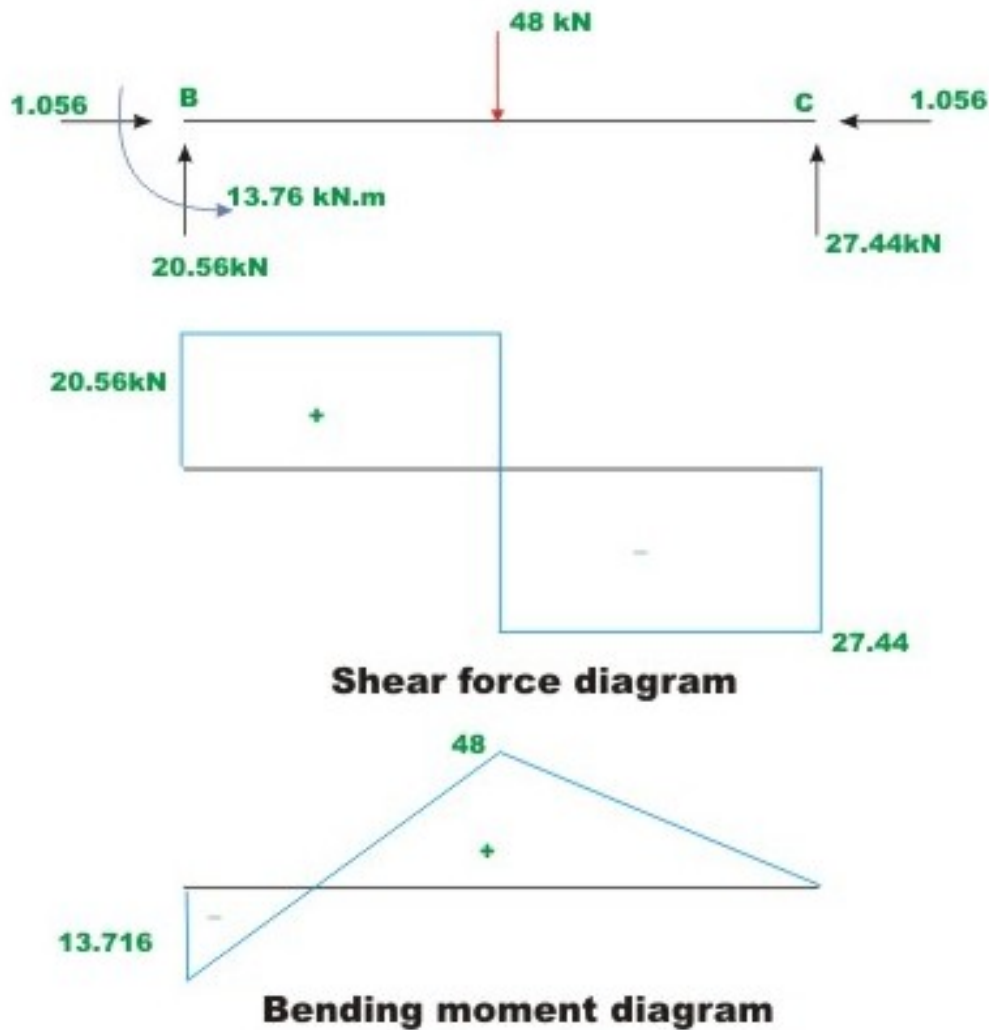


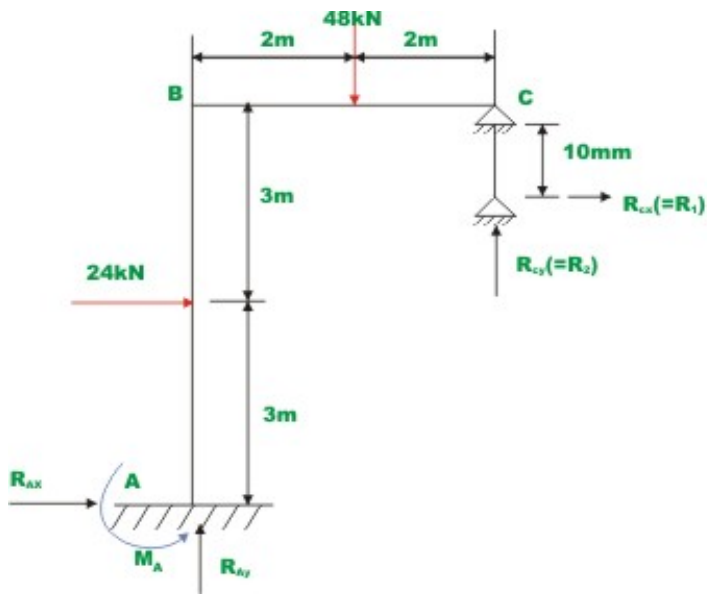
Fig. 11.2f

## 11.2 Support settlements

As discussed in the case of statically indeterminate beams, the reactions are induced in the case of indeterminate frame due to yielding of supports even when there are no external loads acting on it. The yielding of supports may be either linear displacements or rotations of supports (only in the case of fixed supports). The compatibility condition is that the total displacement of the determinate frame (primary structure) due to external loading and that due to redundant reaction at a given support must be equal to the predicted amount of yielding at that support. If the support is unyielding then it must be equal to zero.

### Example 11.3

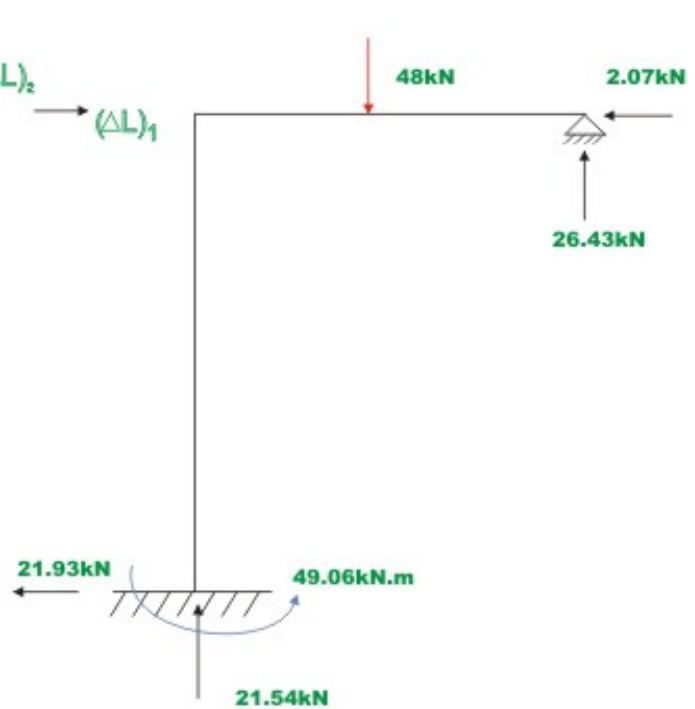
A rigid frame  $ABC$  is loaded as shown in the Fig 11.3a, Compute the reactions if the support  $D$  settles by  $10\text{ mm}$ . vertically downwards. Assume  $EI$  to be constant for all members. Assume  $E = 200\text{ GPa}$  and  $I = 10^{-4}\text{ m}^2$ .



**Fig 11.3(a) Example 11.3**



**Fig 11.3 (b) Primary Structure**



**Fig 11.3 (c) Reactions**

This problem is similar to the previous example except for the support settlement. Hence only change will be in the compatibility equations. The released structure is as shown in Fig.11.3b. The deflections  $(\Delta_L)_1$  and  $(\Delta_L)_2$  at C in the primary structure due to external loading has already been computed in the previous example. Hence,

$$(\Delta_L)_1 = \frac{2052}{EI} \quad (1)$$

$$(\Delta_L)_2 = \frac{-3296}{EI} \quad (2)$$

Therefore,

$$(\Delta_L)_1 = 0.1026 \text{ m}$$

$$(\Delta_L)_2 = -0.1635 \text{ m}$$

The flexibility coefficients are,

$$a_{11} = \frac{72}{EI} \quad (3)$$

$$a_{12} = a_{21} = \frac{-72}{EI} \quad (4)$$

$$a_{22} = \frac{117.33}{EI} \quad (5)$$

Now, the compatibility equations may be written as,

$$(\Delta_L)_1 + a_{11}R_1 + a_{12}R_2 = 0$$

$$(\Delta_L)_2 + a_{21}R_1 + a_{22}R_2 = -10 \times 10^{-3} \quad (6)$$

Solving which,

$$R_1 = -2.072 \text{ kN (towards left)}$$

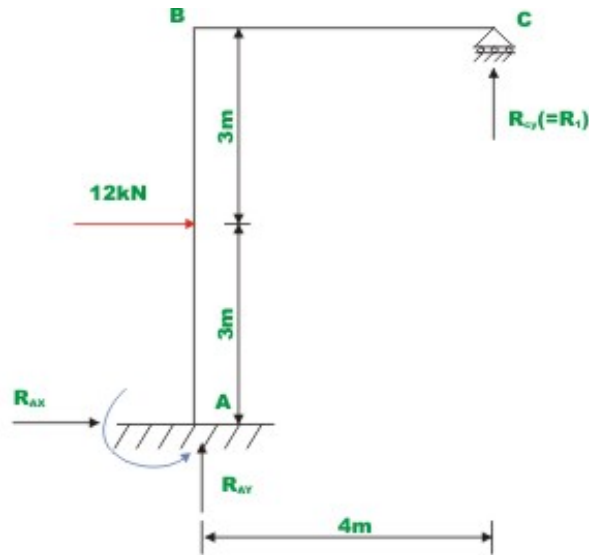
$$R_2 = +26.4 \text{ kN (upwards)} \quad (7)$$

The reactions are shown in Fig.11.3c.

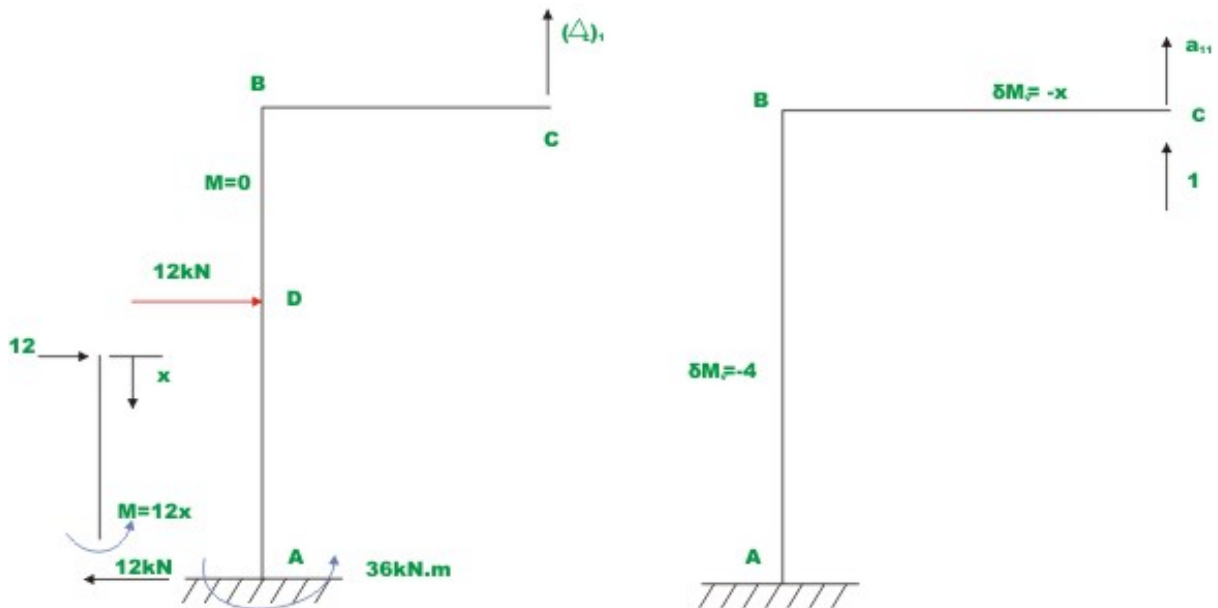


### Example 11.4

Compute the reactions of the rigid frame shown in Fig.11.4a and draw bending moment diagram .Also sketch the deformed shape of the frame. Assume  $EI$  to be constant for all members.



**Fig 11.4 (a) Example 11.4**



**Fig 11.4 (b) Primary structure with external load**

**Fig 11.4 ( c ) Primary structure with unit load along  $R_1$**

Select vertical reaction at C,  $R_1$  as the redundant .Releasing constraint against redundant, the primary structure is obtained. It is shown in Fig.11.4b.

The deflection  $(\Delta_L)_1$  in the primary structure due to external loading can be calculated from unit load method.

$$\begin{aligned}
 (\Delta_L)_1 &= \int_0^3 \frac{(12x)(-4)dx}{EI} \\
 &\quad \text{(span DA, origin at D)} \\
 &= \frac{-216}{EI} \text{ (Downwards)} \qquad (1)
 \end{aligned}$$

Now, compute the flexibility coefficient,

$$\begin{aligned}
 a_{11} &= \int_0^4 \frac{x^2}{EI} dx + \int_0^6 \frac{16}{EI} dx \\
 &= \frac{117.33}{EI} \qquad (2)
 \end{aligned}$$

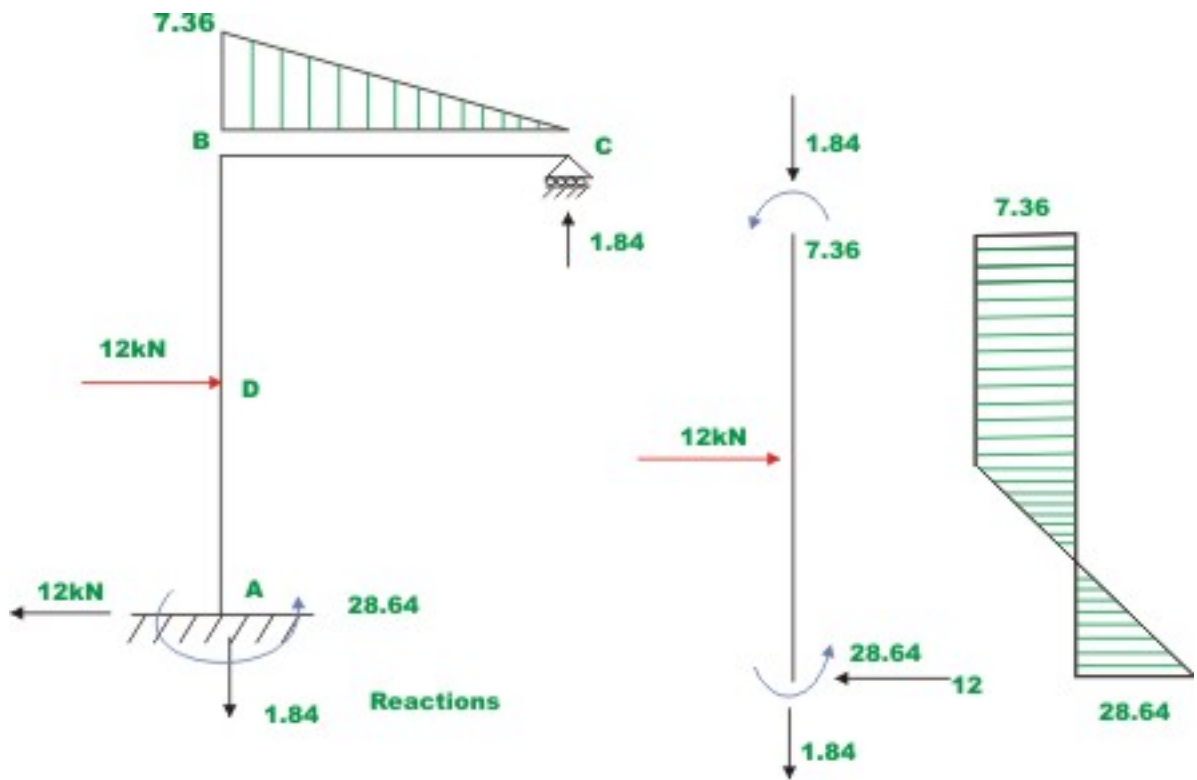
The compatibility condition at support C is that the displacement at C in the primary structure due to external loading plus the displacement at C due to redundant must vanish. Thus,

$$\frac{-216}{EI} + \frac{117.33}{EI} R_1 = 0 \qquad (3)$$

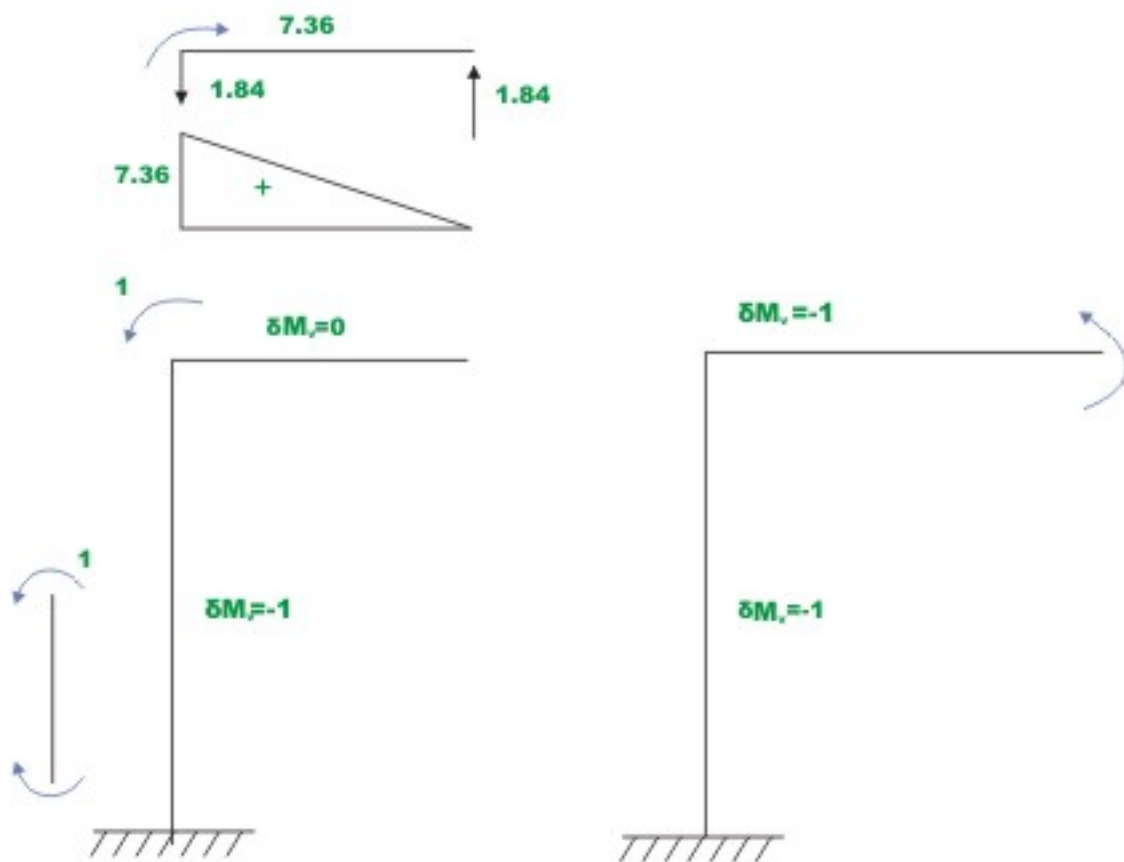
Solving,

$$R_1 = 1.84 \text{ kN} \qquad (4)$$

The remaining reactions are calculated from static equilibrium equations. They are shown in Fig.11.4d along with the bending moment diagram.



**Fig 11.4 (d) bending moment diagram plotted on Compression side**



**Fig 11.4(e)**

**Fig 11.4(f)**

To sketch the deformed shape/elastic curve of the frame, it is required to compute rotations of joints  $B$  and  $C$  and horizontal displacement of  $C$ . These joint rotations and displacements can also be calculated from the principle of superposition. The joint rotations are taken to be positive when clockwise. Towards this end first calculate joint rotations at  $B$  ( $\theta_{BL}$ ) and  $C$  ( $\theta_{CL}$ ) and horizontal displacement at  $C$  in the released structure (refer to Fig.11.4b). This can be evaluated by unit load method.

$$\theta_{BL} = \int_0^3 \frac{12(x)(-1)}{EI} dx = \frac{-54}{EI} \quad (5)$$

$$\theta_{CL} = \int_0^3 \frac{(12x)(-1)}{EI} dx = \frac{-54}{EI} \quad (6)$$

$$\Delta_{CL} = \int_0^3 \frac{12x(3+x)}{EI} dx = \frac{270}{EI} \quad (7)$$

Next, calculate the joint rotations and displacements when unit value of redundant is applied (Fig.11.4c). Let the joint rotations and displacements be  $\theta_{BR}$ ,  $\theta_{CR}$  and  $\Delta_{CR}$ .

$$\theta_{BR} = \int_0^6 \frac{4dx}{EI} = \frac{24}{EI} \quad (8)$$

$$\theta_{CR} = \int_0^4 \frac{(-x)(-1)}{EI} dx + \int_0^6 \frac{(-4)(-1)}{EI} dx = \frac{32}{EI} \quad (9)$$

$$\Delta_{CR} = \int_0^6 \frac{(-4)x}{EI} dx = \frac{-72}{EI} \quad (10)$$

Now using the principle of superposition, the actual rotations and displacements at the joints may be obtained.

$$\begin{aligned} \theta_B &= \theta_{BL} + \theta_{BR} R_1 \\ &= \frac{-54}{EI} + \frac{24 \times 1.84}{EI} = \frac{-9.84}{EI} \end{aligned} \quad (11)$$

(Clockwise rotation)

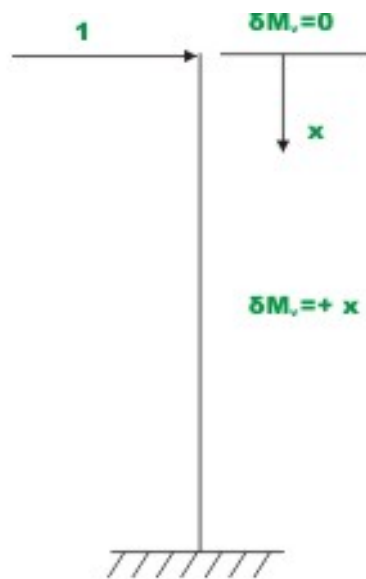
$$\begin{aligned} \theta_C &= \theta_{CL} + \theta_{CR} \times R_1 \\ &= \frac{-54}{EI} + \frac{32 \times 1.84}{EI} = \frac{4.88}{EI} \end{aligned} \quad (12)$$

(Counterclockwise rotation)

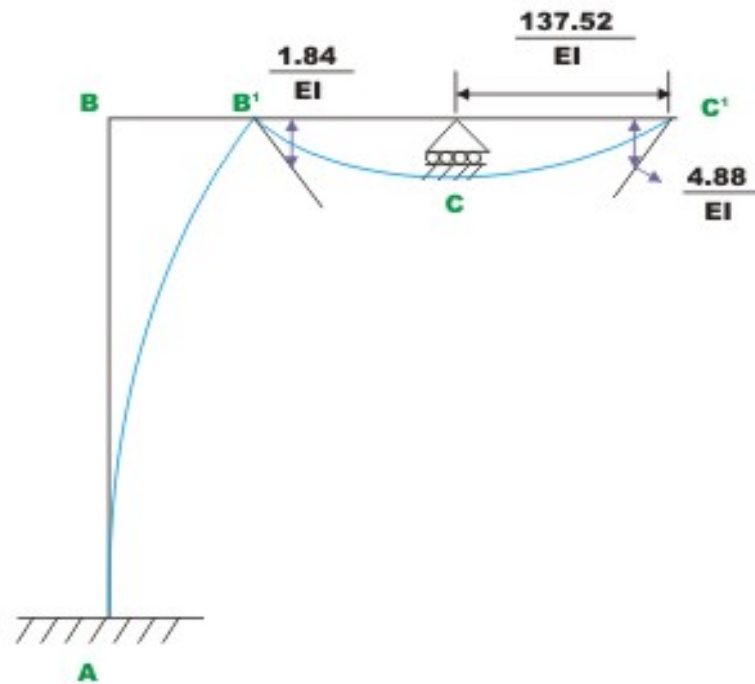
$$\Delta_C = \Delta_{CL} + \Delta_{CR} R_1 \quad (13)$$

$$= \frac{270}{EI} - \frac{72 \times 1.84}{EI} = \frac{137.52}{EI} \quad (\text{towards right})$$

The qualitative elastic curve is shown in Fig. 11.4h.



**Fig 11.4 (g)**



**Fig 11.4 (h) Deformed shape**

## Summary

In this lesson, the statically indeterminate plane frames are analysed by force method. For the purpose of illustrations only bending deformations of the frame are considered as the axial deformations are very small. The problem of yielding of supports in the case of plane frames is also discussed. The procedure to draw qualitative elastic curve of the frame is illustrated with the help of typical example. The bending moment and shear force diagrams are also drawn for the case of plane frame.