Module
1
Energy Methods in Structural Analysis

Lesson 6

Engesser's Theorem and Truss Deflections by Virtual Work Principles

Instructional Objectives

After reading this lesson, the reader will be able to:

1. State and prove Crotti-Engesser theorem.
2. Derive simple expressions for calculating deflections in trusses subjected to mechanical loading using unit-load method.
3. Derive equations for calculating deflections in trusses subjected to temperature loads.
4. Compute deflections in trusses using unit-load method due to fabrication errors.

6.1 Introduction

In the previous lesson, we discussed the principle of virtual work and principle of virtual displacement. Also, we derived unit – load method from the principle of virtual work and unit displacement method from the principle of virtual displacement. In this lesson, the unit load method is employed to calculate displacements of trusses due to external loading. Initially the Engesser's theorem, which is more general than the Castigliano's theorem, is discussed. In the end, few examples are solved to demonstrate the power of virtual work.

6.2 Crotti-Engesser Theorem

The Crotti-Engesser theorem states that the first partial derivative of the complementary strain energy (U^*) expressed in terms of applied forces F_j is equal to the corresponding displacement.

$$\frac{\partial U^*}{\partial F_j} = \sum_{k=1}^{n} a_{jk} F_k = u_j \tag{6.1}$$

For the case of indeterminate structures this may be stated as,

$$\frac{\partial U^*}{\partial F_j} = 0 \tag{6.2}$$

Note that Engesser's theorem is valid for both linear and non-linear structures. When the complementary strain energy is equal to the strain energy (i.e. in case of linear structures) the equation (6.1) is nothing but the statement of Castigliano's first theorem in terms of complementary strain energy.

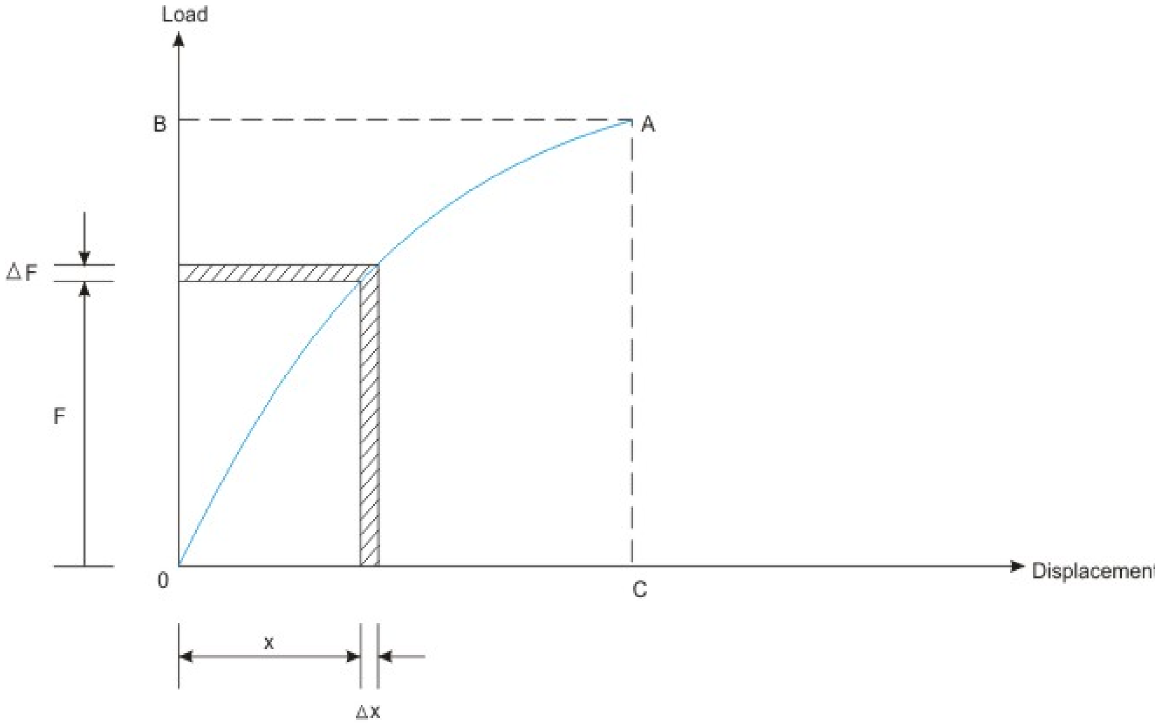

Fig. 6.1 Non-linear Load-displacement curve.

In the above figure the strain energy (area OACO) is not equal to complementary strain energy (area OABO)

$$Area \;\; OACO \; = U = \int_0^u F \, du \tag{6.3}$$

Differentiating strain energy with respect to displacement,

$$\frac{dU}{du} = F \tag{6.4}$$

This is the statement of Castigliano's second theorem. Now the complementary energy is equal to the area enclosed by OABO.

$$U^* = \int_0^F u \;\, dF \tag{6.5}$$

Differentiating complementary strain energy with respect to force F ,

$$\frac{dU^*}{dF} = u \tag{6.6}$$

This gives deflection in the direction of load. When the load displacement relationship is linear, the above equation coincides with the Castigliano's first theorem given in equation (3.8).

6.3 Unit Load Method as applied to Trusses

6.3.1 External Loading

In case of a plane or a space truss, the only internal forces present are axial as the external loads are applied at joints. Hence, equation (5.7) may be written as,

$$\sum_{j=1}^{n} \delta F_j u_j = \int_0^L \frac{\delta P_v P ds}{EA} \tag{6.7}$$

wherein, δF_j is the external virtual load, u_j are the actual deflections of the truss, δP_v is the virtual stress resultant in the frame due to the virtual load and $\int_0^L \frac{P}{EA} ds$ is the actual internal deformation of the frame due to real forces. In the above equation L, E, A respectively represent length of the member, cross-sectional area of a member and modulus of elasticity of a member. In the unit load method, $\delta F_j = 1$ and all other components of virtual forces $\delta F_i \ (i = 1, 2, ..., j-1, j+1, ..., n)$ are zero. Also, if the cross sectional area A of truss remains constant throughout, then integration may be replaced by summation and hence equation (6.7) may be written as,

$$u_j = \sum_{i=1}^{m} \frac{(\delta P_v)_{ij} P_i L_i}{E_i A_i} \tag{6.8}$$

where m is the number of members, $(\delta P_v)_{ij}$ is the internal virtual axial force in member i due to unit virtual load at j and $(\frac{P_i}{E_i A_i}) L_i$ is the total deformation of member i due to real loads. If we represent total deformation by Δ_i, then

$$u_j = \sum_{i=1}^{m} (\delta P_v)_{ij} \Delta_i \tag{6.9}$$

where, Δ_i is the true change in length of member i due to real loads.

6.3.2 Temperature Loading

Due to change in the environmental temperature, the truss members either expand or shrink. This in turn produces joint deflections in the truss. This may be calculated by equation (6.9). In this case, the change in length of member Δ_i is calculated from the relation,

$$\Delta_i = \alpha T L_i \tag{6.10}$$

where α is the co-efficient of thermal expansion member, L_i is the length of member and T is the temperature change.

6.3.3 Fabrication Errors and Camber

Sometimes, there will be errors in fabricating truss members. In some cases, the truss members are fabricated slightly longer or shorter in order to provide camber to the truss. Usually camber is provided in bridge truss so that its bottom chord is curved upward by an equal to its downward deflection of the chord when subjected to dead. In such instances, also, the truss joint deflection is calculated by equation (6.9). Here,

$$\Delta_i = e_i \tag{6.11}$$

where, e_i is the fabrication error in the length of the member. e_i is taken as positive when the member lengths are fabricated slightly more than the actual length otherwise it is taken as negative.

6.4 Procedure for calculating truss deflection

1. First, calculate the real forces in the member of the truss either by method of joints or by method of sections due to the externally applied forces. From this determine the actual deformation (Δ_i) in each member from the equation $\frac{P_i L_i}{E_i A_i}$. Assume tensile forces as positive and compressive forces as negative.

2. Now, consider the virtual load system such that only a unit load is considered at the joint either in the horizontal or in the vertical direction, where the deflection is sought. Calculate virtual forces $(\delta P_v)_{ij}$ in each member due to the applied unit load at the *j*-th joint.

3. Now, using equation (6.9), evaluate the *j*-th joint deflection u_j.

4. If deflection of a joint needs to be calculated due to temperature change, then determine the actual deformation (Δ_i) in each member from the equation $\Delta_i = \alpha T L_i$.

The application of equation (6.8) is shown with the help of few problems.

Example 6.1

Find horizontal and vertical deflection of joint C of truss ABCD loaded as shown in Fig. 6.2a. Assume that, all members have the same axial rigidity.

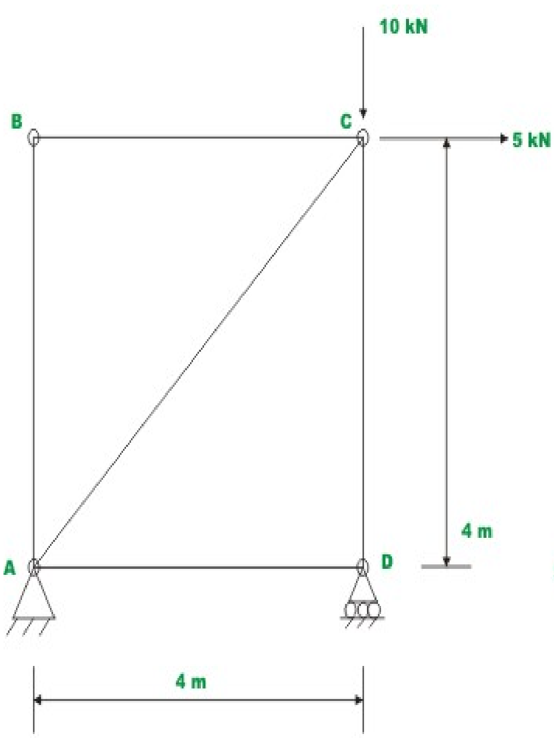

Fig. 6.2a Example 6.1

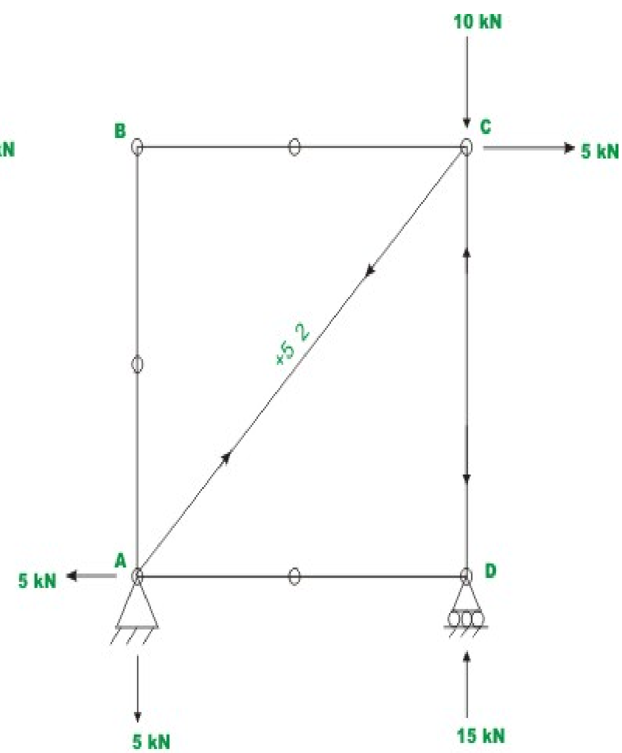

Fig. 6.2b Reaction and forces in members.

The given truss is statically determinate one. The reactions are as shown in Fig 6.2b along with member forces which are determined by equations of static equilibrium. To evaluate horizontal deflection at 'C', apply a unit load as shown in Fig 6.2c and evaluate the virtual forces δP_v in each member. The magnitudes of internal forces are also shown in the respective figures. The tensile forces are shown as +ve and compressive forces are shown as –ve. At each end of the bar, arrows have been drawn indicating the direction in which the force in the member acts on the joint.

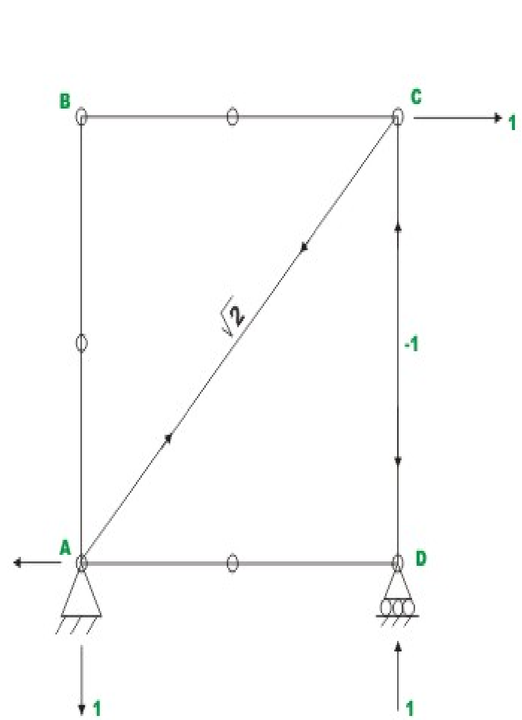

Fig. 6.2c Reaction and member forces due to vertical horizontal force at C.

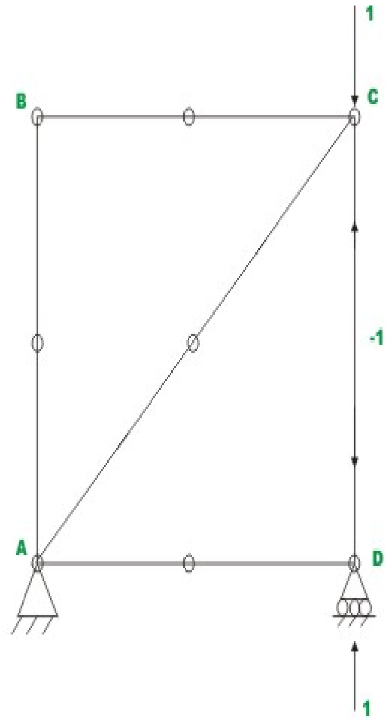

Fig. 6.2d Reaction and member forces due to vertical horizontal force at C.

Horizontal deflection at joint C is calculated with the help of unit load method. This may be stated as,

$$1 \times u_c^H = \sum \frac{(\delta P_v)_{ic} P_i L_i}{E_i A_i} \quad (1)$$

For calculating horizontal deflection at C, u_c, apply a unit load at the joint C as shown in Fig.6.2c. The whole calculations are shown in table 6.1. The calculations are self explanatory.

Table 6.1 Computational details for horizontal deflection at C

Member	Length	$L_i / A_i E_i$	P_i	$(\delta P_v)_i$	$\frac{(\delta P_v)_i P_i L_i}{E_i A_i}$
units	m	m/kN	kN	kN	kN.m
AB	4	4/AE	0	0	0
BC	4	4/AE	0	0	0
CD	4	4/AE	-15	-1	60/AE
DA	4	4/AE	0	0	0
AC	$4\sqrt{2}$	$4\sqrt{2}$ /AE	$5\sqrt{2}$	$\sqrt{2}$	$40\sqrt{2}$ /AE
				$\sum$	$\frac{60+40\sqrt{2}}{AE}$

$$(1)(u_C^H) \rightarrow = \frac{60+40\sqrt{2}}{AE} = \frac{116.569}{AE} \qquad \text{(Towards right)} \qquad (2)$$

Vertical deflection at joint C

$$1 \times u_c^v = \sum \frac{(\delta P_v^v)_{ic} P_i L_i}{E_i A_i} \qquad (3)$$

In this case, a unit vertical load is applied at joint C of the truss as shown in Fig. 6.2d.

Table 6.2 Computational details for vertical deflection at C

Member	Length	$L_i / A_i E_i$	P_i	$(\delta P_v^v)_i$	$\frac{(\delta P_v)_i P_i L_i}{E_i A_i}$
units	m	m/kN	kN	kN	kN.m
AB	4	4/AE	0	0	0
BC	4	4/AE	0	0	0
CD	4	4/AE	-15	-1	60/AE
DA	4	4/AE	0	0	0
AC	$4\sqrt{2}$	$4\sqrt{2}$ /AE	$5\sqrt{2}$	0	0
				$\sum$	$\frac{60}{AE}$

$$(1)(u_C^v) \downarrow = \frac{60}{AE} = \frac{60}{AE} \qquad \text{(Downwards)} \qquad (4)$$

Example 6.2

Compute the vertical deflection of joint b and horizontal displacement of joint D of the truss shown in Fig. 6.3a due to

a) Applied loading as shown in figure.

b) Increase in temperature of 25^0 C in the top chord BD. Assume $\alpha = \frac{1}{75000}$ $per°C$, $E = 2.00 \times 10^5$ N/mm^2. The cross sectional areas of the members in square centimeters are shown in parentheses.

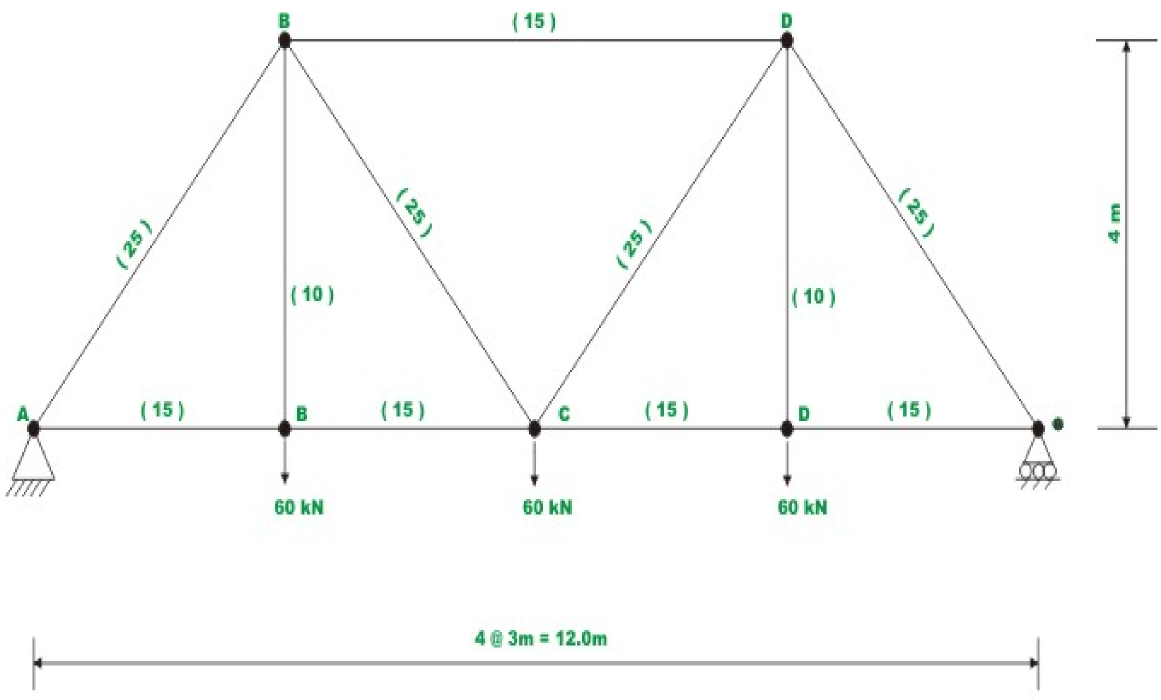

Fig. 6.3a. Example 6.2

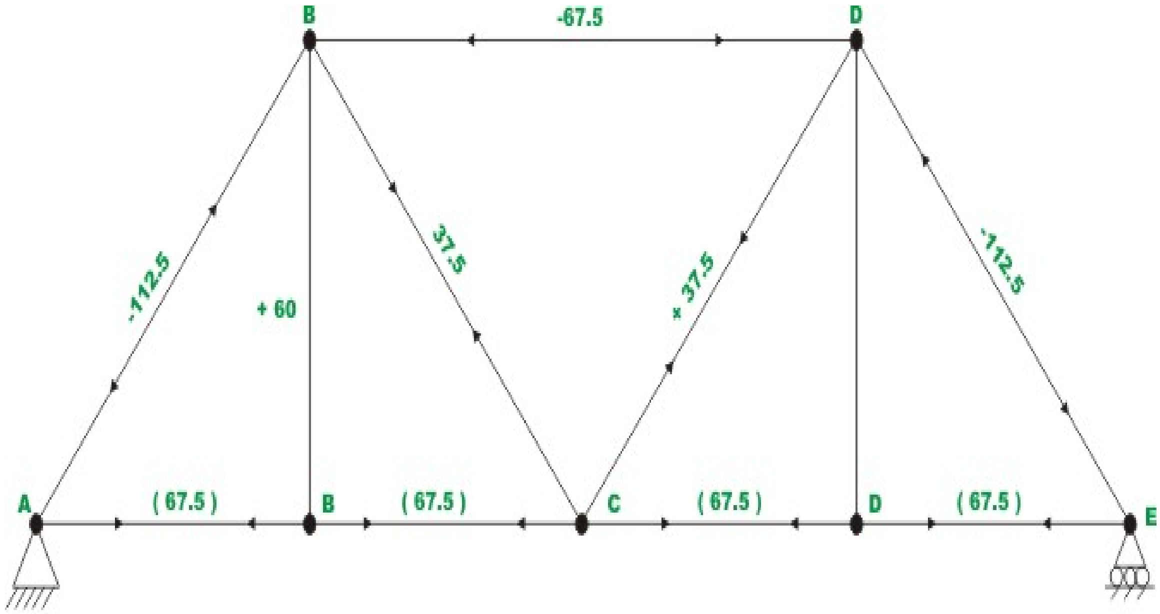

Fig. 6.3b Reaction and member forces due to applied load

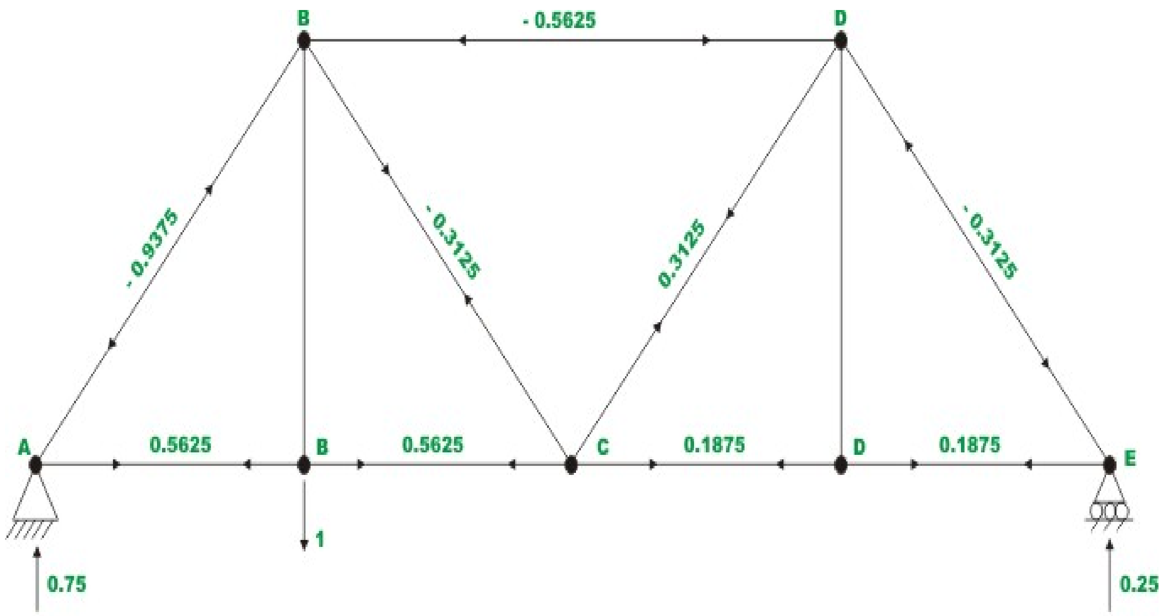

Fig. 6.3c Forces in members due to unit virtual vertical force at b.

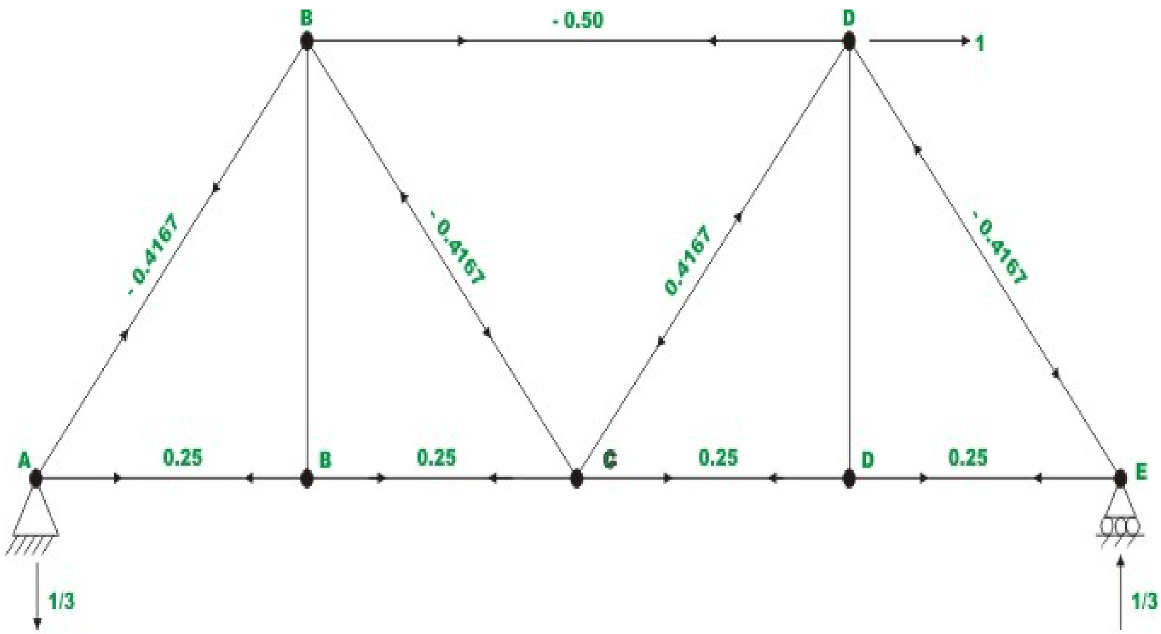

Fig. 6.3d Forces in members due to unit horizontal force at D.

The complete calculations are shown in the following table.

Table 6.3 Computational details for example 6.2

Mem	L_i	$L_i / A_i E_i$	P_i	$(\delta P_v^v)_i$	$(\delta P_v^H)_i$	$\Delta_{ti} = \alpha t L_i$	$\frac{(\delta P_v^v)_i P_i L_i}{E_i A_i}$	$\frac{(\delta P_v^H)_i P_i L_i}{E_i A_i}$	$(\delta P_v^v)_i \Delta_{ti}$	$(\delta P_v^H)_i \Delta_{ti}$
units	m	(10^{-5}) m/kN	kN	kN	kN	m	(10^{-3}) kN.m	(10^{-3}) kN.m	(10^{-3}) kN.m	(10^{-3}) kN.m
aB	5	1.0	-112.5	-0.937	+0.416	0	1.05	-0.47	0	0
ab	3	1.0	+67.5	+0.562	+0.750	0	0.38	0.51	0	0
bc	3	1.0	+67.5	+0.562	+0.750	0	0.38	0.51	0	0
Bc	5	1.0	+37.5	-0.312	-0.416	0	-0.12	-0.16	0	0
BD	6	2.0	-67.5	-0.562	+0.500	0.002	0.76	-0.68	-1.13	1
cD	5	1.0	+37.5	+0.312	+0.416	0	0.12	0.16	0	0
cd	3	1.0	+67.5	+0.187	+0.250	0	0.13	0.17	0	0
de	3	1.0	+67.5	+0.187	+0.250	0	0.13	0.17	0	0
De	5	1.0	-112.5	-0.312	-0.416	0	0.35	0.47	0	0
Bb	4	2.0	+60.0	1	0	0	1.2	0	0	0
Dd	4	2.0	+60.0	0	0	0	0	0	0	0
						$\sum$	4.38	0.68	-1.13	1

a) Vertical deflection of joint b

Applying principle of virtual work as applied to an ideal pin jointed truss,

$$\sum_{j=1}^{1} \delta F_j u_j = \sum_{i=1}^{m} \frac{(\delta P_v)_{ij} P_i L_i}{E_i A_i} \qquad (1)$$

For calculating vertical deflection at b, apply a unit virtual load $\delta F_b = 1$. Then the above equation may be written as,

$$1 \times u_b^v = \sum \frac{(\delta P_v^v)_i P_i L_i}{E_i A_i} \qquad (2)$$

1) Due to external loads

$$u_b \downarrow = \frac{+0.00438 \ \ KNm}{1 \ KN} = 0.00438 \ \ m$$

$$= 4.38 \ \ mm \downarrow$$

2) Due to change in temperature

$$(1)(u_b^t \downarrow) = \sum (\delta P_v^v)_i \Delta_{ti}$$

$$u_b^t \downarrow = \frac{-0.001125 \ \ KN.m}{1 \ \ KN} = -0.00113 m$$

$$u_b^t = 1.13 \ \ mm \uparrow$$

b) Horizontal displacement of joint 'D'

1) Due to externally applied loads

$$1 \times u_b^H = \sum \frac{(\delta P_v^H)_i P_i L_i}{E_i A_i}$$

$$u_D^H \rightarrow = \frac{+0.00068 \ \ KNm}{1 \ KN} = 0.00068 \ \ m$$

$$= 0.68 \ \ mm \rightarrow$$

2) Due to change in temperature

$$(1)(u_D^{Ht} \rightarrow) = \sum (\delta P_v^H)_i \Delta_{ti}$$

$$u_D^{Ht} \rightarrow = \frac{0.001 \ KN.m}{1 \ KN} = 0.001m$$

$$u_D^{Ht} = 1.00 \ mm \rightarrow$$

Summary

In this chapter the Crotti-Engessor's theorem which is more general than the Castigliano's theorem has been introduced. The unit load method is applied statically determinate structure for calculating deflections when the truss is subjected to various types of loadings such as: mechanical loading, temperature loading and fabrication errors.