Module
1
Energy Methods in Structural Analysis

Lesson
4
Theorem of Least Work

Instructional Objectives

After reading this lesson, the reader will be able to:

1. State and prove theorem of Least Work.
2. Analyse statically indeterminate structure.
3. State and prove Maxwell-Betti's Reciprocal theorem.

4.1 Introduction

In the last chapter the Castigliano's theorems were discussed. In this chapter theorem of least work and reciprocal theorems are presented along with few selected problems. We know that for the statically determinate structure, the partial derivative of strain energy with respect to external force is equal to the displacement in the direction of that load at the point of application of load. This theorem when applied to the statically indeterminate structure results in the theorem of least work.

4.2 Theorem of Least Work

According to this theorem, the partial derivative of strain energy of a statically indeterminate structure with respect to statically indeterminate action should vanish as it is the function of such redundant forces to prevent any displacement at its point of application. The forces developed in a redundant framework are such that the total internal strain energy is a minimum. This can be proved as follows. Consider a beam that is fixed at left end and roller supported at right end as shown in Fig. 4.1a. Let $P_1, P_2, \ldots, P_n$ be the forces acting at distances $x_1, x_2, \ldots, x_n$ from the left end of the beam of span L. Let $u_1, u_2, \ldots, u_n$ be the displacements at the loading points $P_1, P_2, \ldots, P_n$ respectively as shown in Fig. 4.1a. This is a statically indeterminate structure and choosing R_a as the redundant reaction, we obtain a simple cantilever beam as shown in Fig. 4.1b. Invoking the principle of superposition, this may be treated as the superposition of two cases, viz, a cantilever beam with loads $P_1, P_2, \ldots, P_n$ and a cantilever beam with redundant force R_a (see Fig. 4.2a and Fig. 4.2b)

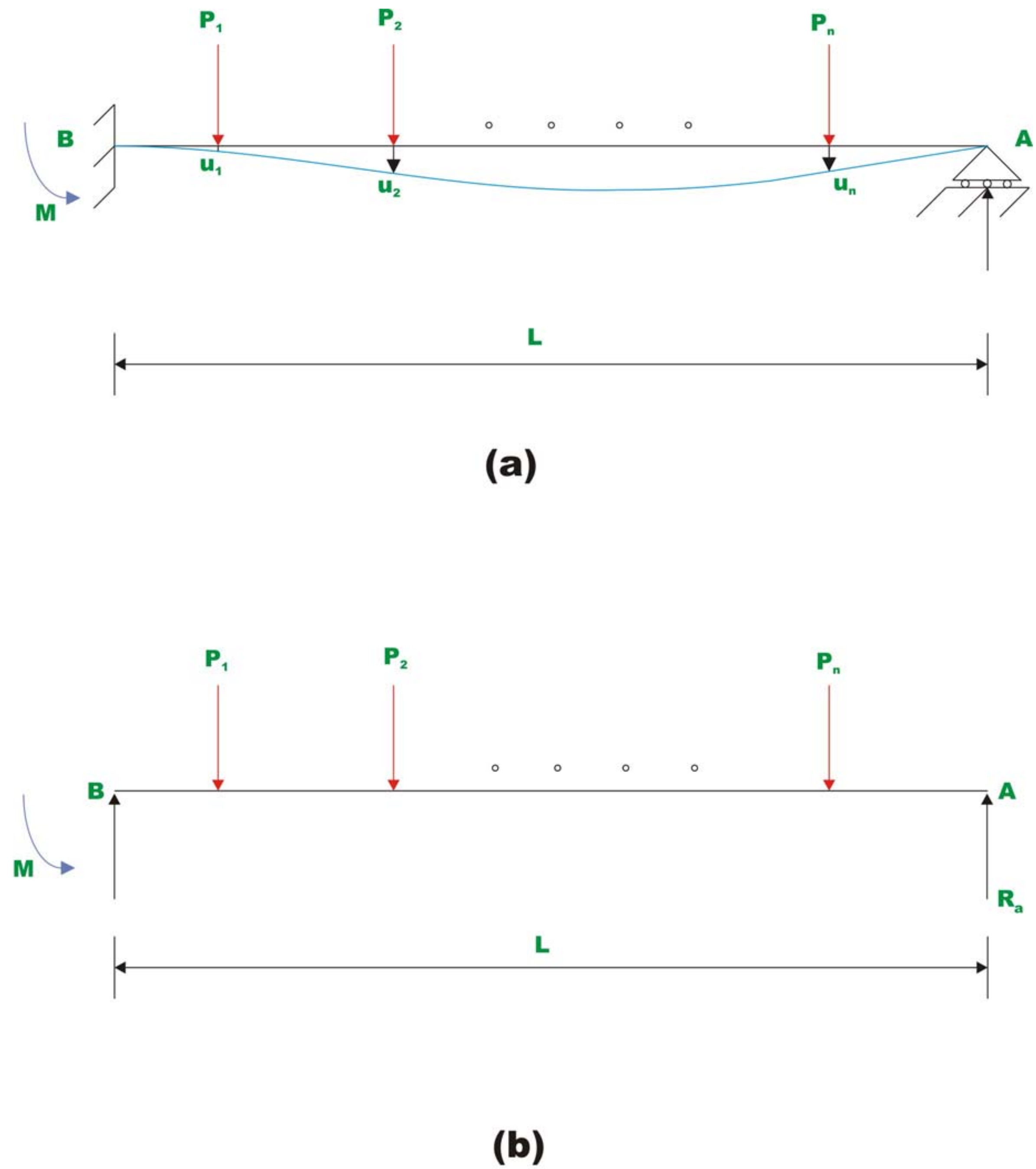


Fig.4.1 Theorem of Least work

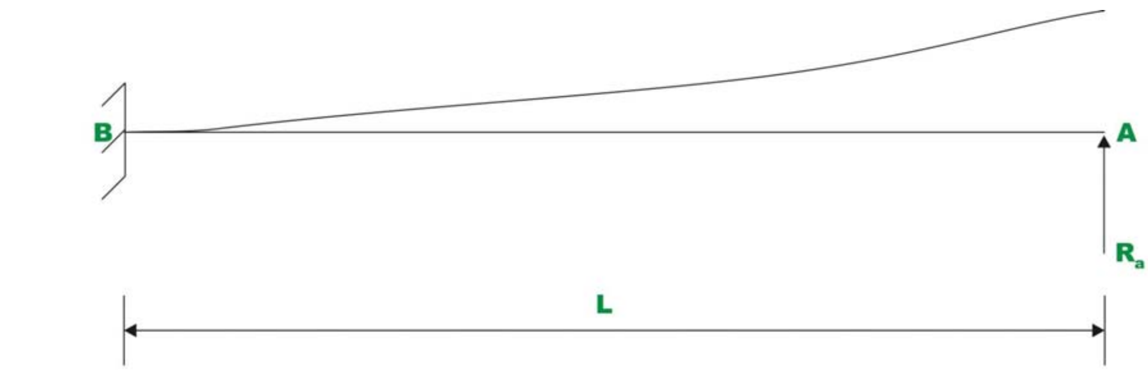


Fig. 4.2 (a)Cantilever beam with redundant

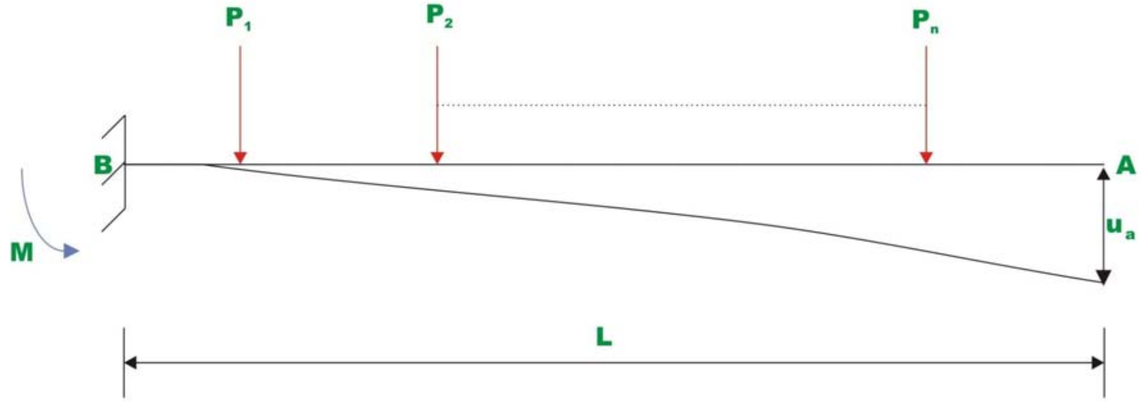


Fig. 4.2 (b)Cantilever beam with externally applied loads and a fictious load

In the first case (4.2a), obtain deflection below A due to applied loads $P_1, P_2, ..., P_n$. This can be easily accomplished through Castigliano's first theorem as discussed in Lesson 3. Since there is no load applied at A, apply a fictitious load Q at A as in Fig. 4.2. Let u_a be the deflection below A.

Now the strain energy U_s stored in the determinate structure (i.e. the support A removed) is given by,

$$U_S = \frac{1}{2}P_1u_1 + \frac{1}{2}P_2u_2 + ... + \frac{1}{2}P_nu_n + \frac{1}{2}Qu_a \qquad (4.1)$$

It is known that the displacement u_1 below point P_1 is due to action of $P_1, P_2, ..., P_n$ acting at $x_1, x_2, ..., x_n$ respectively and due to Q at A. Hence, u_1 may be expressed as,

$$u_1 = a_{11}P_1 + a_{12}P_2 + ... + a_{1n}P_n + a_{1a}Q \qquad (4.2)$$

where, a_{ij} is the flexibility coefficient at i due to unit force applied at j. Similar equations may be written for $u_2, u_3, ..., u_n$ and u_a. Substituting for $u_2, u_3, ..., u_n$ and u_a in equation (4.1) from equation (4.2), we get,

$$U_S = \frac{1}{2}P_1[a_{11}P_1 + a_{12}P_2 + ... + a_{1n}P_n + a_{1a}Q] + \frac{1}{2}P_2[a_{21}P_1 + a_{22}P_2 + ...a_{2n}P_n + a_{2a}Q] + ...$$
$$+ \frac{1}{2}P_n[a_{n1}P_1 + a_{n2}P_2 + ...a_{nn}P_n + a_{na}Q] + \frac{1}{2}Q[a_{a1}P_1 + a_{a2}P_2 + ... + a_{an}P_n + a_{aa}Q] \qquad (4.3)$$

Taking partial derivative of strain energy U_s with respect to Q, we get deflection at A.

$$\frac{\partial U_s}{\partial Q} = a_{a1}P_1 + a_{a2}P_2 + ... + a_{an}P_n + a_{aa}Q \qquad (4.4)$$

Substitute $Q = 0$ as it is fictitious in the above equation,

$$\frac{\partial U_s}{\partial Q} = u_a = a_{a1}P_1 + a_{a2}P_2 + ... + a_{an}P_n \qquad (4.5)$$

Now the strain energy stored in the beam due to redundant reaction R_A is,

$$U_r = \frac{R_a^2 L^3}{6EI} \qquad (4.6)$$

Now deflection at A due to R_a is

$$\frac{\partial U_r}{\partial R_a} = -u_a = \frac{R_a L^3}{3EI} \qquad (4.7)$$

The deflection due to R_a should be in the opposite direction to one caused by superposed loads $P_1, P_2, ..., P_n$, so that the net deflection at A is zero. From equation (4.5) and (4.7) one could write,

$$\frac{\partial Us}{\partial Q} = u_a = -\frac{\partial U_r}{\partial R_a} \qquad (4.8)$$

Since Q is fictitious, one could as well replace it by R_a. Hence,

$$\frac{\partial}{\partial R_a}(U_s + U_r) = 0 \qquad (4.9)$$

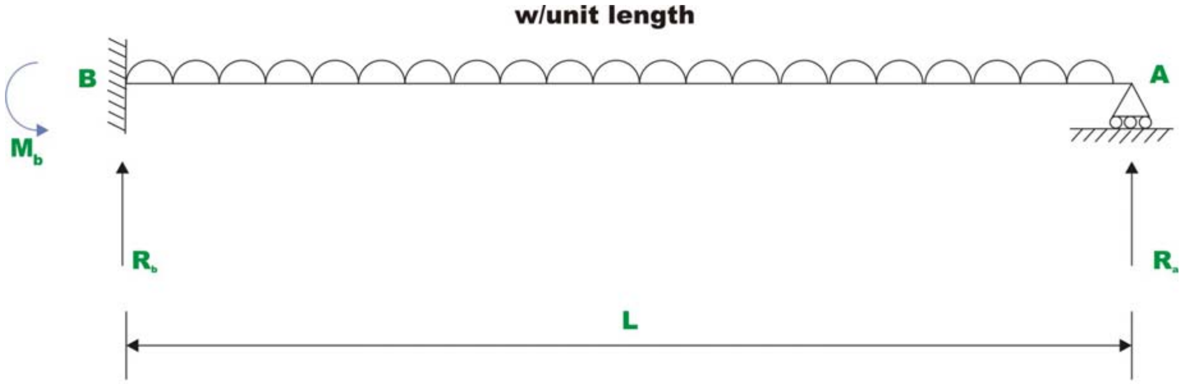
or,

$$\frac{\partial U}{\partial R_a} = 0 \qquad (4.10)$$

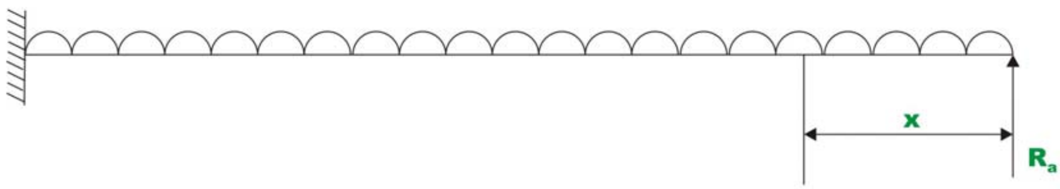
This is the statement of theorem of least work. Where U is the total strain energy of the beam due to superimposed loads $P_1, P_2, ..., P_n$ and redundant reaction R_a.

Example 4.1

Find the reactions of a propped cantilever beam uniformly loaded as shown in Fig. 4.3a. Assume the flexural rigidity of the beam EI to be constant throughout its length.



(a)



(b)

Fig.4.3 Example 4.1

There three reactions R_a, R_b and M_b as shown in the figure. We have only two equation of equilibrium viz., $\sum F_y = 0$ and $\sum M = 0$. This is a statically indeterminate structure and choosing R_b as the redundant reaction, we obtain a simple cantilever beam as shown in Fig. 4.3b.
Now, the internal strain energy of the beam due to applied loads and redundant reaction, considering only bending deformations is,

$$U = \int_0^L \frac{M^2}{2EI} dx \quad (1)$$

According to theorem of least work we have,

$$\frac{\partial U}{\partial R_b} = 0 = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial R_b} \quad (2)$$

Bending moment at a distance x from B, $M = R_b x - \dfrac{wx^2}{2}$ (3)

$$\frac{\partial M}{\partial R_b} = x \quad (4)$$

Hence,

$$\frac{\partial U}{\partial R_b} = \int_0^L \frac{(R_b x - wx^2/2)x}{EI}dx \quad (5)$$

$$\frac{\partial U}{\partial R_b} = \left[\frac{R_B L^3}{3} - \frac{wL^4}{8}\right]\frac{1}{EI} = 0 \quad (6)$$

Solving for R_b, we get,

$$R_B = \frac{3}{8}wL$$

$$R_a = wL - R_b = \frac{5}{8}wL \text{ and } M_a = -\frac{wL^2}{8} \quad (7)$$

Example 4.2

A ring of radius R is loaded as shown in figure. Determine increase in the diameter AB of the ring. Young's modulus of the material is E and second moment of the area is I about an axis perpendicular to the page through the centroid of the cross section.

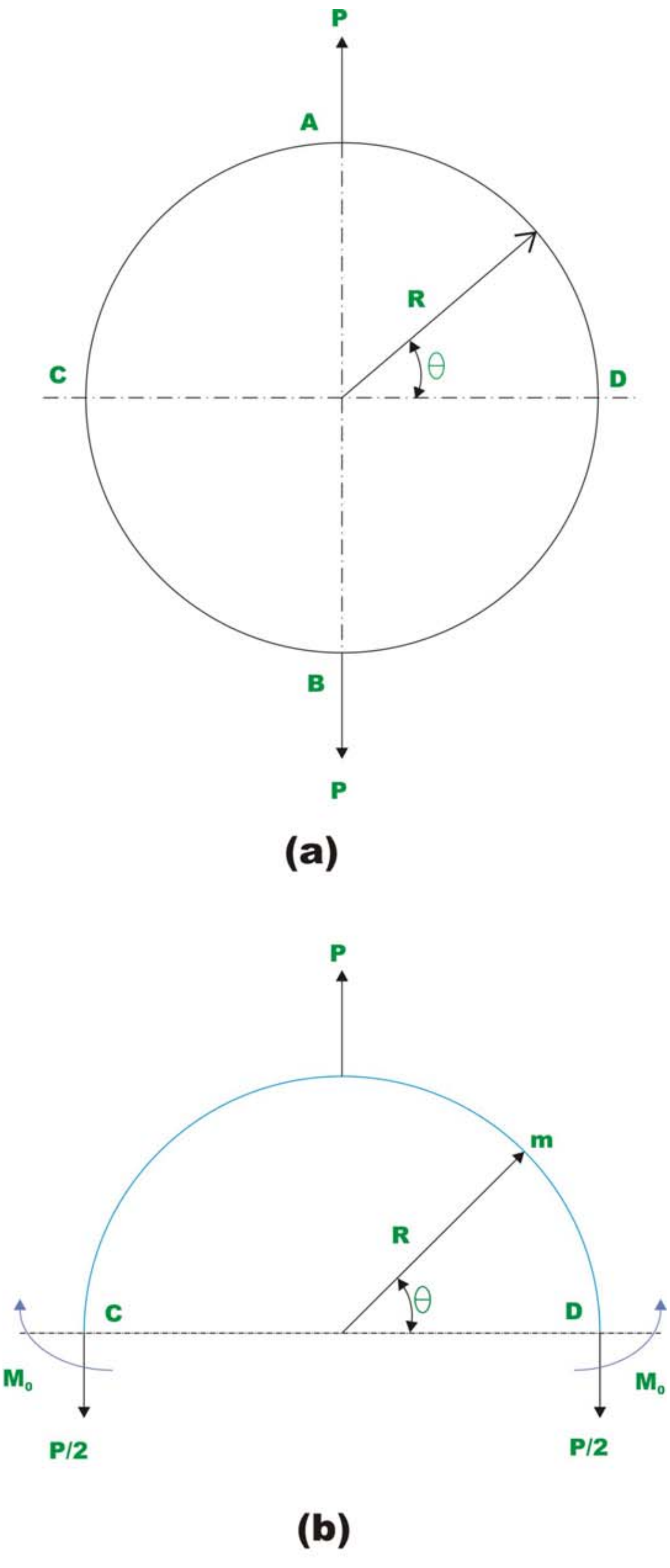


Fig.4.4 Example 4.2

The free body diagram of the ring is as shown in Fig. 4.4. Due to symmetry, the slopes at C and D is zero. The value of redundant moment M_0 is such as to make slopes at C and D zero. The bending moment at any section θ of the beam is,

$$M = M_0 - \frac{PR}{2}(1-\cos\theta) \tag{1}$$

Now strain energy stored in the ring due to bending deformations is,

$$U = \int_0^{2\pi} \frac{M^2 R}{2EI} d\theta \tag{2}$$

Due to symmetry, one could consider one quarter of the ring. According to theorem of least work,

$$\frac{\partial U}{\partial M_0} = 0 = \int_0^{2\pi} \frac{M}{EI} \frac{\partial M}{\partial M_0} R d\theta \tag{3}$$

$$\frac{\partial M}{\partial M_0} = 1$$

$$\frac{\partial U}{\partial M_0} = \int_0^{2\pi} \frac{M}{EI} R d\theta \tag{4}$$

$$0 = \frac{4R}{EI} \int_0^{\frac{\pi}{2}} [M_0 - \frac{PR}{2}(1-\cos\theta)]\, d\theta \tag{5}$$

Integrating and solving for M_0,

$$M_0 = PR\left(\frac{1}{2} - \frac{1}{\pi}\right) \tag{6}$$

$$M_0 = 0.182PR$$

Now, increase in diameter Δ, may be obtained by taking the first partial derivative of strain energy with respect to P. Thus,

$$\Delta = \frac{\partial U}{\partial P}$$

Now strain energy stored in the ring is given by equation (2). Substituting the value of M_0 and equation (1) in (2), we get,

$$U = \frac{2R}{EI}\int_0^{\pi/2}\{\frac{PR}{2}(\frac{2}{\pi}-1)-\frac{PR}{2}(1-\cos\theta)\}^2 d\theta \tag{7}$$

Now the increase in length of the diameter is,

$$\frac{\partial U}{\partial P} = \frac{2R}{EI}\int_0^{\pi/2} 2\{\frac{PR}{2}(\frac{2}{\pi}-1)-\frac{PR}{2}(1-\cos\theta)\}\{\frac{R}{2}(\frac{2}{\pi}-1)-\frac{R}{2}(1-\cos\theta)\}d\theta \tag{8}$$

After integrating,

$$\Delta = \frac{PR^3}{EI}\{\frac{\pi}{4}-\frac{2}{\pi}\} = 0.149\frac{PR^3}{EI} \tag{9}$$

4.3 Maxwell–Betti Reciprocal theorem

Consider a simply supported beam of span L as shown in Fig. 4.5. Let this beam be loaded by two systems of forces P_1 and P_2 separately as shown in the figure. Let u_{21} be the deflection below the load point P_2 when only load P_1 is acting. Similarly let u_{12} be the deflection below load P_1, when only load P_2 is acting on the beam.

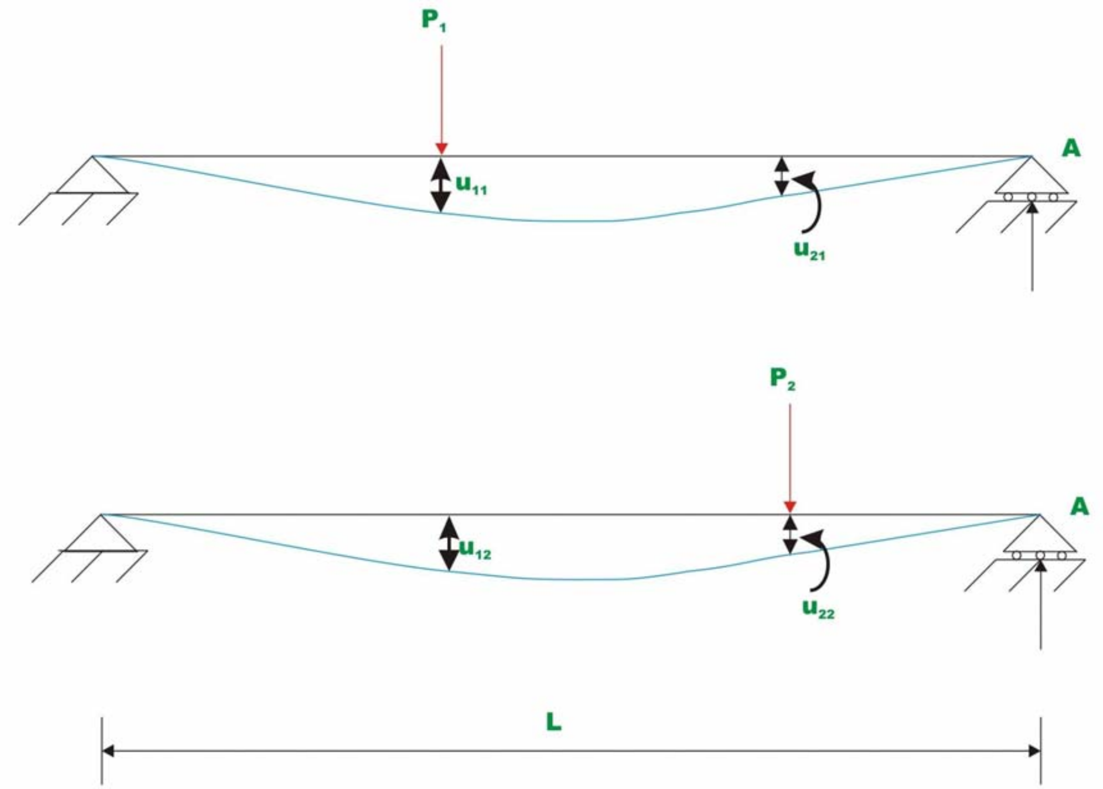


Fig. 4.5 Reciprocal theorem

The reciprocal theorem states that the work done by forces acting through displacement of the second system is the same as the work done by the second system of forces acting through the displacements of the first system. Hence, according to reciprocal theorem,

$$P_1 \times u_{12} = P_2 \times u_{21} \tag{4.11}$$

Now, u_{12} and u_{21} can be calculated using Castiglinao's first theorem. Substituting the values of u_{12} and u_{21} in equation (4.27) we get,

$$P_1 \times \frac{5P_2L^3}{48EI} = P_2 \times \frac{5P_1L^3}{48EI} \tag{4.12}$$

Hence it is proved. This is also valid even when the first system of forces is $P_1, P_2, ..., P_n$ and the second system of forces is given by $Q_1, Q_2, ..., Q_n$. Let $u_1, u_2, ..., u_n$ be the displacements caused by the forces $P_1, P_2, ..., P_n$ only and $\delta_1, \delta_2, ..., \delta_n$ be the displacements due to system of forces $Q_1, Q_2, ..., Q_n$ only acting on the beam as shown in Fig. 4.6.

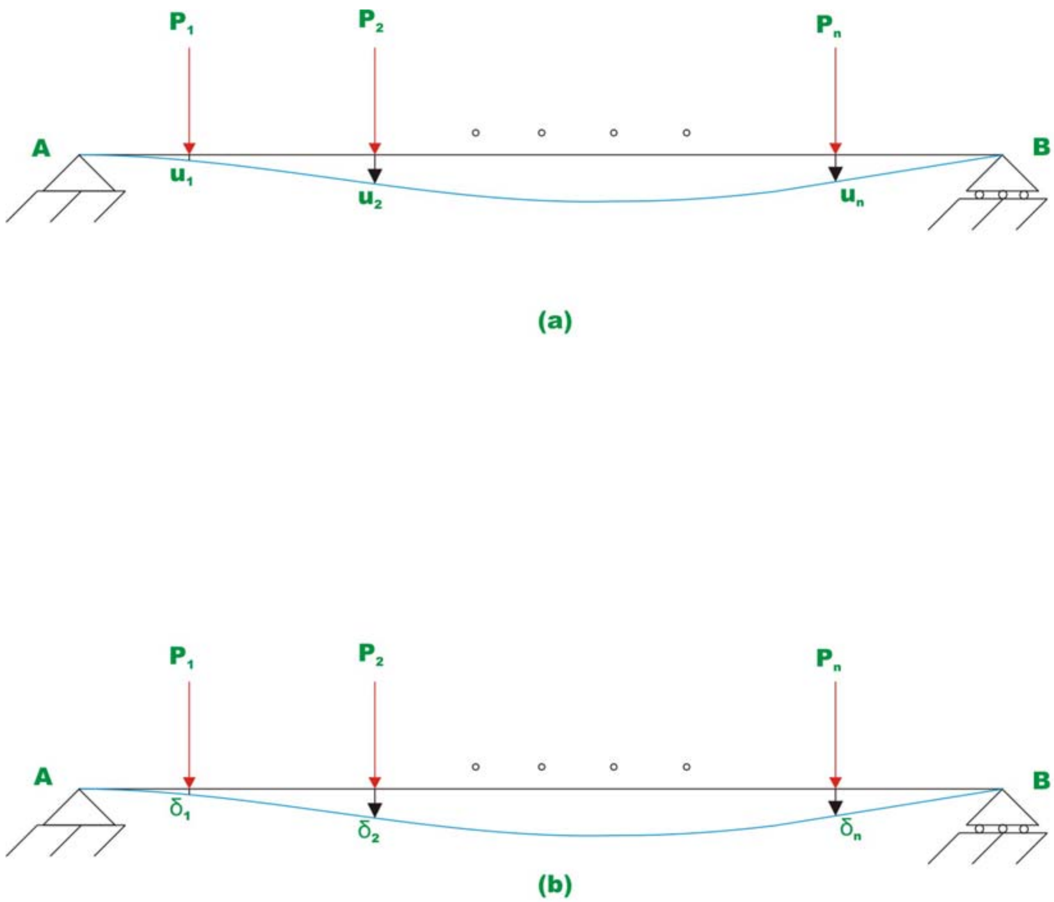


Fig. 4.6 Generalized statement of Reciprocal Theorem

Now the reciprocal theorem may be stated as,

$$P_i \, \delta_i = Q_i \, u_i \qquad i = 1,2,....,n \qquad (4.13)$$

Summary

In lesson 3, the Castigliano's first theorem has been stated and proved. For statically determinate structure, the partial derivative of strain energy with respect to external force is equal to the displacement in the direction of that load at the point of application of the load. This theorem when applied to the statically indeterminate structure results in the theorem of Least work. In this chapter the theorem of Least Work has been stated and proved. Couple of problems is solved to illustrate the procedure of analysing statically indeterminate structures. In the end, the celebrated theorem of Maxwell-Betti's reciprocal theorem has been sated and proved.