Module
1
Energy Methods in Structural Analysis

Lesson 2

Principle of Superposition, Strain Energy

Instructional Objectives

After reading this lesson, the student will be able to

1. State and use principle of superposition.
2. Explain strain energy concept.
3. Differentiate between elastic and inelastic strain energy and state units of strain energy.
4. Derive an expression for strain energy stored in one-dimensional structure under axial load.
5. Derive an expression for elastic strain energy stored in a beam in bending.
6. Derive an expression for elastic strain energy stored in a beam in shear.
7. Derive an expression for elastic strain energy stored in a circular shaft under torsion.

2.1 Introduction

In the analysis of statically indeterminate structures, the knowledge of the displacements of a structure is necessary. Knowledge of displacements is also required in the design of members. Several methods are available for the calculation of displacements of structures. However, if displacements at only a few locations in structures are required then energy based methods are most suitable. If displacements are required to solve statically indeterminate structures, then only the relative values of EA, EI and GJ are required. If actual value of displacement is required as in the case of settlement of supports and temperature stress calculations, then it is necessary to know actual values of E and G. In general deflections are small compared with the dimensions of structure but for clarity the displacements are drawn to a much larger scale than the structure itself. Since, displacements are small, it is assumed not to cause gross displacements of the geometry of the structure so that equilibrium equation can be based on the original configuration of the structure. When non-linear behaviour of the structure is considered then such an assumption is not valid as the structure is appreciably distorted. In this lesson two of the very important concepts i.e., principle of superposition and strain energy method will be introduced.

2.2 Principle of Superposition

The principle of superposition is a central concept in the analysis of structures. This is applicable when there exists a linear relationship between external forces and corresponding structural displacements. The principle of superposition may be stated as *the deflection at a given point in a structure produced by several loads acting simultaneously on the structure can be found by superposing deflections at the same point produced by* loads *acting individually*. This is illustrated with the help of a simple beam problem. Now consider a cantilever beam of length L and having constant flexural rigidity EI subjected to two externally applied forces P_1 and P_2 as shown in Fig. 2.1. From moment-area theorem we can evaluate deflection below C, which states that the tangential deviation of point c from the tangent at point A is equal to the first moment of the area of the $\frac{M}{EI}$ diagram between A and C about C. Hence, the deflection u below C due to loads P_1 and P_2 acting simultaneously is (by moment-area theorem),

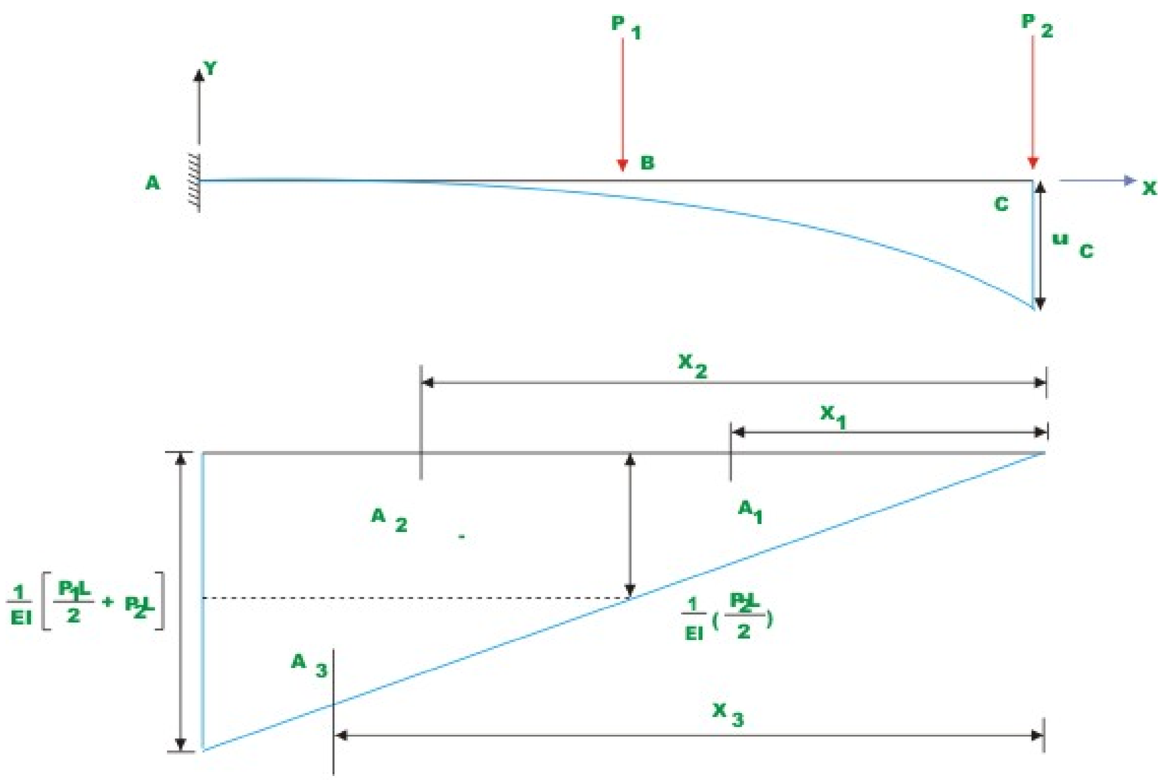

Fig 2.1 Cantilever Beam with Two Concentrated Loads

$$u = A_1 \bar{x}_1 + A_2 \bar{x}_2 + A_3 \bar{x}_3 \qquad (2.1)$$

where u is the tangential deviation of point C with respect to a tangent at A. Since, in this case the tangent at A is horizontal, the tangential deviation of point

C is nothing but the vertical deflection at C. $\bar{x}_1, \bar{x}_2$ and $\bar{x}_3$ are the distances from point C to the centroids of respective areas respectively.

$$\bar{x}_1 = \frac{2}{3}\frac{L}{2} \qquad \bar{x}_2 = \left(\frac{L}{2} + \frac{L}{4}\right) \qquad \bar{x}_3 = \frac{2}{3}\frac{L}{2} + \frac{L}{2}$$

$$A_1 = \frac{P_2 L^2}{8EI} \qquad A_2 = \frac{P_2 L^2}{4EI} \qquad A_3 = \frac{(P_1 L + P_2 L)L}{8EI}$$

Hence,

$$u = \frac{P_2 L^2}{8EI}\frac{2}{3}\frac{L}{2} + \frac{P_2 L^2}{4EI}\left[\frac{L}{2} + \frac{L}{4}\right] + \frac{(P_1 L + P_2 L)L}{8EI}\left[\frac{2}{3}\frac{L}{2} + \frac{L}{2}\right] \qquad (2.2)$$

After simplification one can write,

$$u = \frac{P_2 L^3}{3EI} + \frac{5 P_1 L^3}{48EI} \qquad (2.3)$$

Now consider the forces being applied separately and evaluate deflection at C in each of the case.

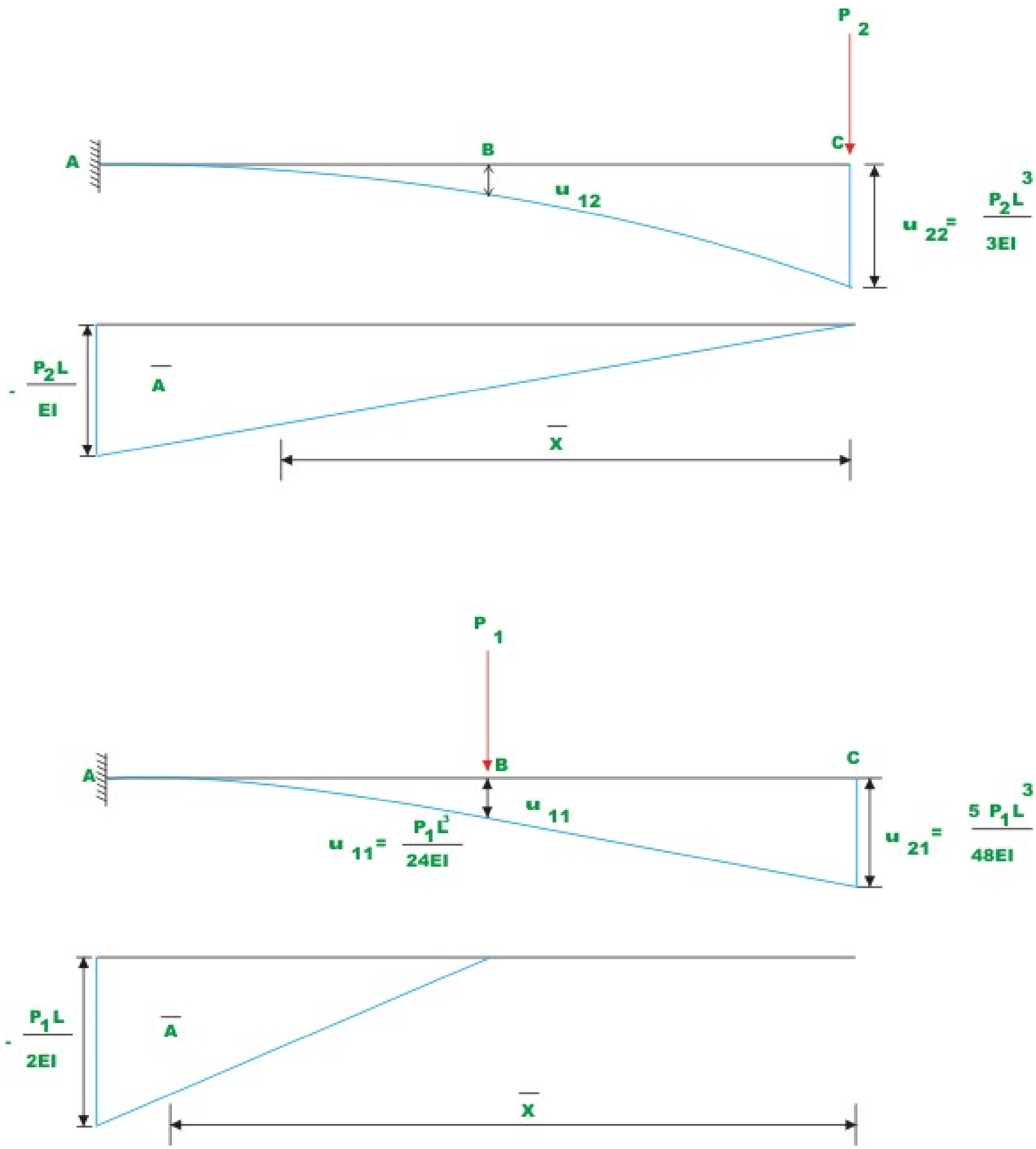

Fig 2.2 Deflection Computation

$$u_{22} = \frac{P_2 L^3}{3EI} \tag{2.4}$$

where u_{22} is deflection at C (2) when load P_1 is applied at C (2) itself. And,

$$u_{21} = \frac{1}{2}\frac{P_1 L}{2EI}\frac{L}{2}\left[\frac{L}{2} + \frac{2}{3}\frac{L}{2}\right] = \frac{5P_1 L^3}{48EI} \qquad (2.5)$$

where u_{21} is the deflection at C (2) when load is applied at $B(1)$. Now the total deflection at C when both the loads are applied simultaneously is obtained by adding u_{22} and u_{21}.

$$u = u_{22} + u_{21} = \frac{P_2 L^3}{3EI} + \frac{5P_1 L^3}{48EI} \qquad (2.6)$$

Hence it is seen from equations (2.3) and (2.6) that when the structure behaves linearly, the total deflection caused by forces $P_1, P_2, ..., P_n$ at any point in the structure is the sum of deflection caused by forces $P_1, P_2, ..., P_n$ acting independently on the structure at the same point. This is known as the Principle of Superposition.

The method of superposition is not valid when the material stress-strain relationship is non-linear. Also, it is not valid in cases where the geometry of structure changes on application of load. For example, consider a hinged-hinged beam-column subjected to only compressive force as shown in Fig. 2.3(a). Let the compressive force P be less than the Euler's buckling load of the structure. Then deflection at an arbitrary point C (say) u_c^1 is zero. Next, the same beam-column be subjected to lateral load Q with no axial load as shown in Fig. 2.3(b). Let the deflection of the beam-column at C be u_c^2. Now consider the case when the same beam-column is subjected to both axial load P and lateral load Q. As per the principle of superposition, the deflection at the centre u_c^3 must be the sum of deflections caused by P and Q when applied individually. However this is not so in the present case. Because of lateral deflection caused by Q, there will be additional bending moment due to P at C. Hence, the net deflection u_c^3 will be more than the sum of deflections u_c^1 and u_c^2.

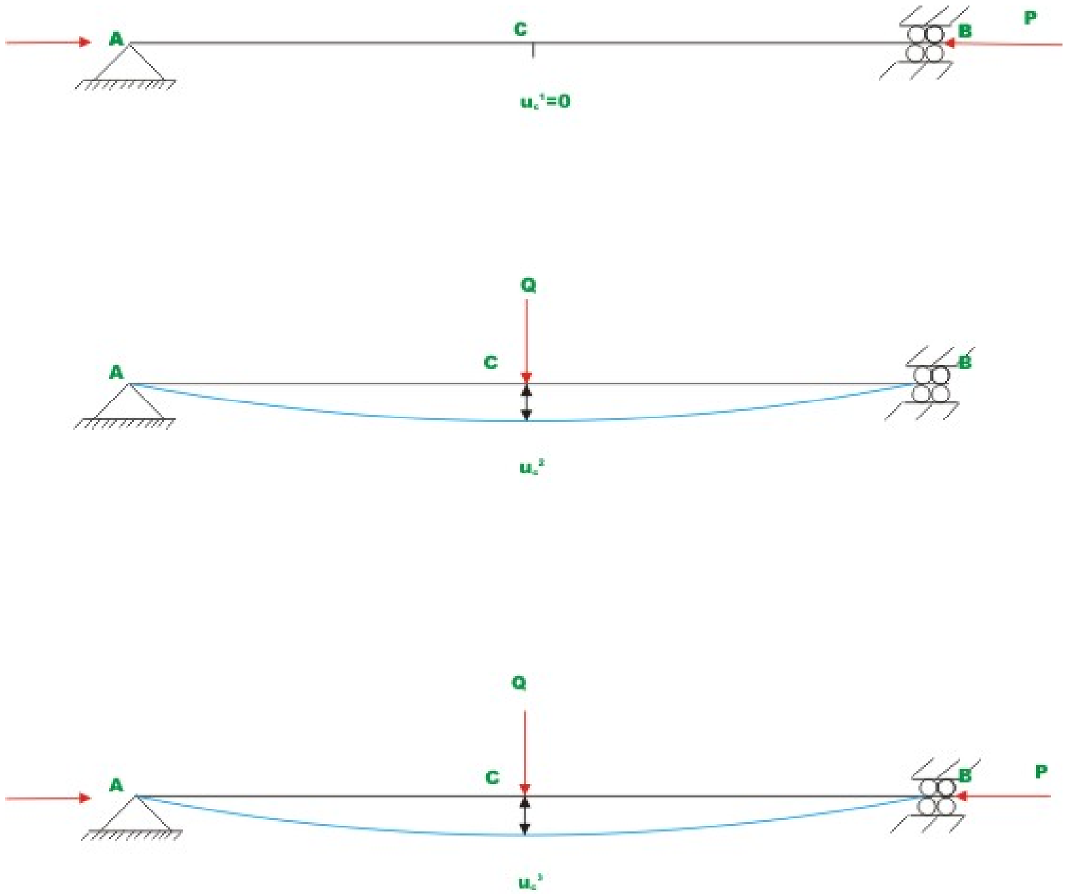

Fig. 2.3

2.3 Strain Energy

Consider an elastic spring as shown in the Fig.2.4. When the spring is slowly pulled, it deflects by a small amount u_1. When the load is removed from the spring, it goes back to the original position. When the spring is pulled by a force, it does some work and this can be calculated once the load-displacement relationship is known. It may be noted that, the spring is a mathematical idealization of the rod being pulled by a force P axially. It is assumed here that the force is applied gradually so that it slowly increases from zero to a maximum value P. Such a load is called static loading, as there are no inertial effects due to motion. Let the load-displacement relationship be as shown in Fig. 2.5. Now, work done by the external force may be calculated as,

$$W_{ext} = \frac{1}{2} P_1 u_1 = \frac{1}{2} (force \times displacement) \tag{2.7}$$

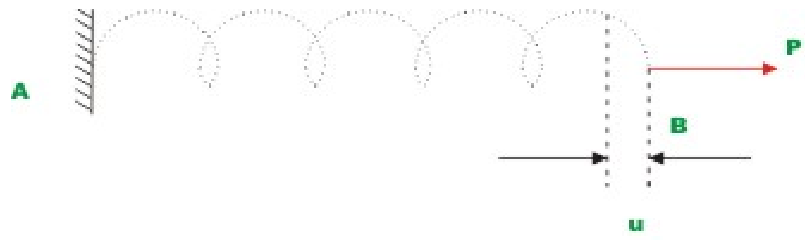

Fig. 2.4 Linear Spring

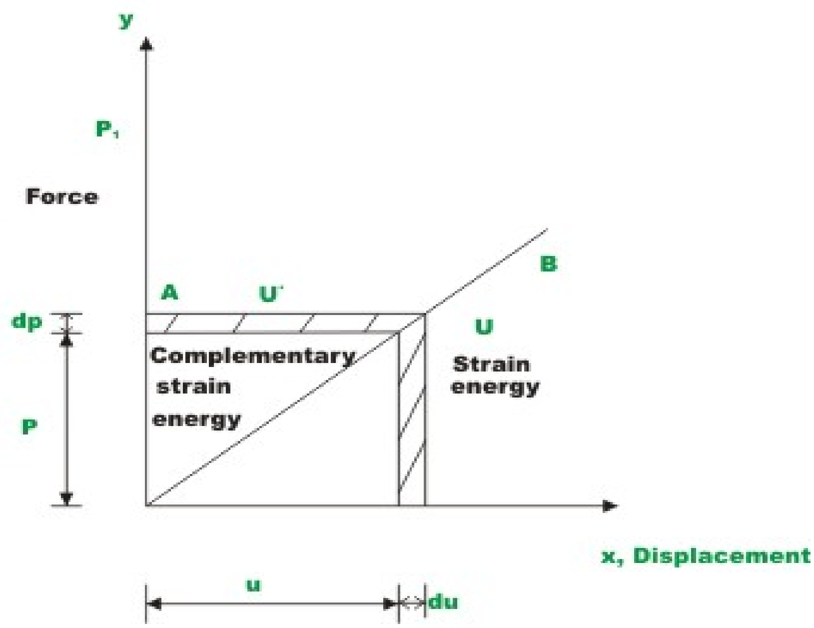

Fig. 2.5 Force-displacement relation

The area enclosed by force-displacement curve gives the total work done by the externally applied load. Here it is assumed that the energy is conserved i.e. the work done by gradually applied loads is equal to energy stored in the structure. This internal energy is known as strain energy. Now strain energy stored in a spring is

$$U = \frac{1}{2} P_1 u_1 \tag{2.8}$$

Work and energy are expressed in the same units. In SI system, the unit of work and energy is the joule (J), which is equal to one Newton metre (N.m). The strain energy may also be defined as the internal work done by the stress resultants in moving through the corresponding deformations. Consider an infinitesimal element within a three dimensional homogeneous and isotropic material. In the most general case, the state of stress acting on such an element may be as shown in Fig. 2.6. There are normal stresses $\left(\sigma_x, \sigma_y \text{ and } \sigma_z\right)$ and shear stresses $\left(\tau_{xy}, \tau_{yz} \text{ and } \tau_{zx}\right)$ acting on the element. Corresponding to normal and shear stresses we have normal and shear strains. Now strain energy may be written as,

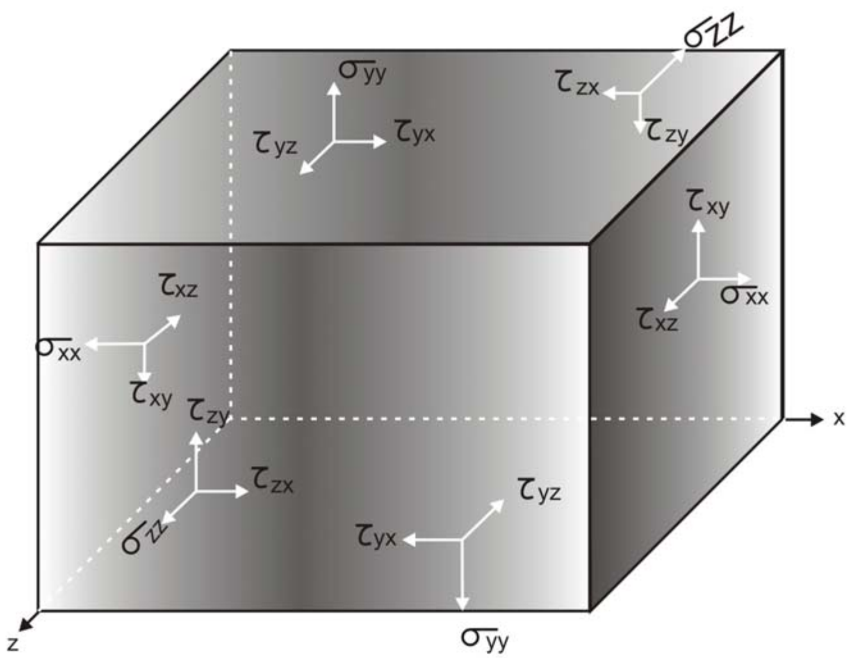

Figure 2.6. Stress on an infinitesimal element .

$$U = \frac{1}{2}\int_v \sigma^T \varepsilon \, dv \tag{2.9}$$

in which σ^T is the transpose of the stress column vector i.e.,

$$\{\sigma\}^T = \left(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}\right) \text{ and } \{\varepsilon\}^T = \left(\varepsilon_x, \varepsilon_y, \varepsilon_z, \varepsilon_{xy}, \varepsilon_{yz}, \varepsilon_{zx}\right) \tag{2.10}$$

The strain energy may be further classified as elastic strain energy and inelastic strain energy as shown in Fig. 2.7. If the force P is removed then the spring shortens. When the elastic limit of the spring is not exceeded, then on removal of load, the spring regains its original shape. If the elastic limit of the material is exceeded, a permanent set will remain on removal of load. In the present case, load the spring beyond its elastic limit. Then we obtain the load-displacement curve $OABCDO$ as shown in Fig. 2.7. Now if at B, the load is removed, the spring gradually shortens. However, a permanent set of OD is till retained. The shaded area BCD is known as the elastic strain energy. This can be recovered upon removing the load. The area $OABDO$ represents the inelastic portion of strain energy.

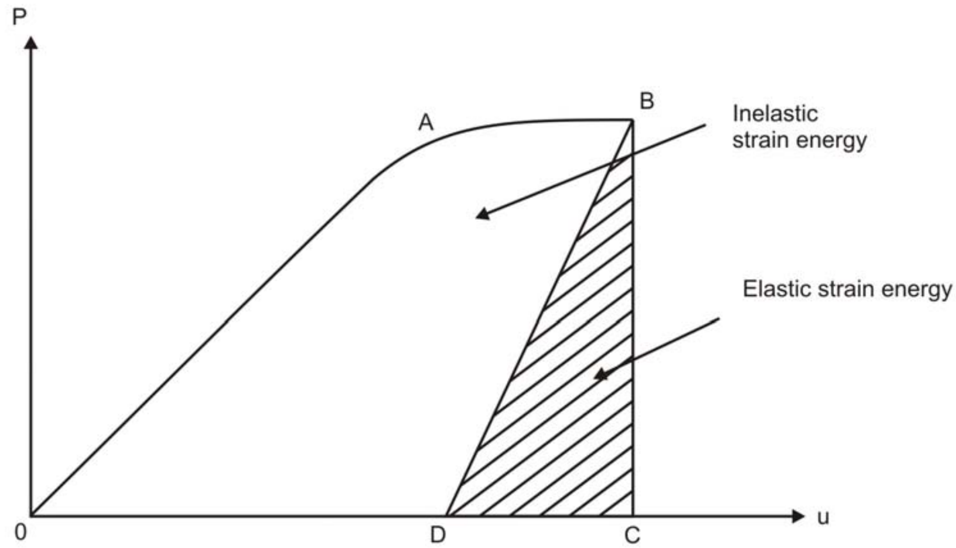

Figure 2.7 Elastic and inelastic strain energy.

The area $OABCDO$ corresponds to strain energy stored in the structure. The area $OABEO$ is defined as the complementary strain energy. For the linearly elastic structure it may be seen that

Area OBC = Area OBE

i.e. Strain energy = Complementary strain energy

This is not the case always as observed from Fig. 2.7. The complementary energy has no physical meaning. The definition is being used for its convenience in structural analysis as will be clear from the subsequent chapters.

Usually structural member is subjected to any one or the combination of bending moment; shear force, axial force and twisting moment. The member resists these external actions by internal stresses. In this section, the internal stresses induced in the structure due to external forces and the associated displacements are calculated for different actions. Knowing internal stresses due to individual forces, one could calculate the resulting stress distribution due to combination of external forces by the method of superposition. After knowing internal stresses and deformations, one could easily evaluate strain energy stored in a simple beam due to axial, bending, shear and torsional deformations.

2.3.1 Strain energy under axial load

Consider a member of constant cross sectional area A, subjected to axial force P as shown in Fig. 2.8. Let E be the Young's modulus of the material. Let the member be under equilibrium under the action of this force, which is applied through the centroid of the cross section. Now, the applied force P is resisted by uniformly distributed internal stresses given by average stress $\sigma = \frac{P}{A}$ as shown by the free body diagram (vide Fig. 2.8). Under the action of axial load P applied at one end gradually, the beam gets elongated by (say) u. This may be calculated as follows. The incremental elongation du of small element of length dx of beam is given by,

$$du = \varepsilon\, dx = \frac{\sigma}{E} dx = \frac{P}{AE} dx \tag{2.11}$$

Now the total elongation of the member of length *L* may be obtained by integration

$$u = \int_0^L \frac{P}{AE} dx \tag{2.12}$$

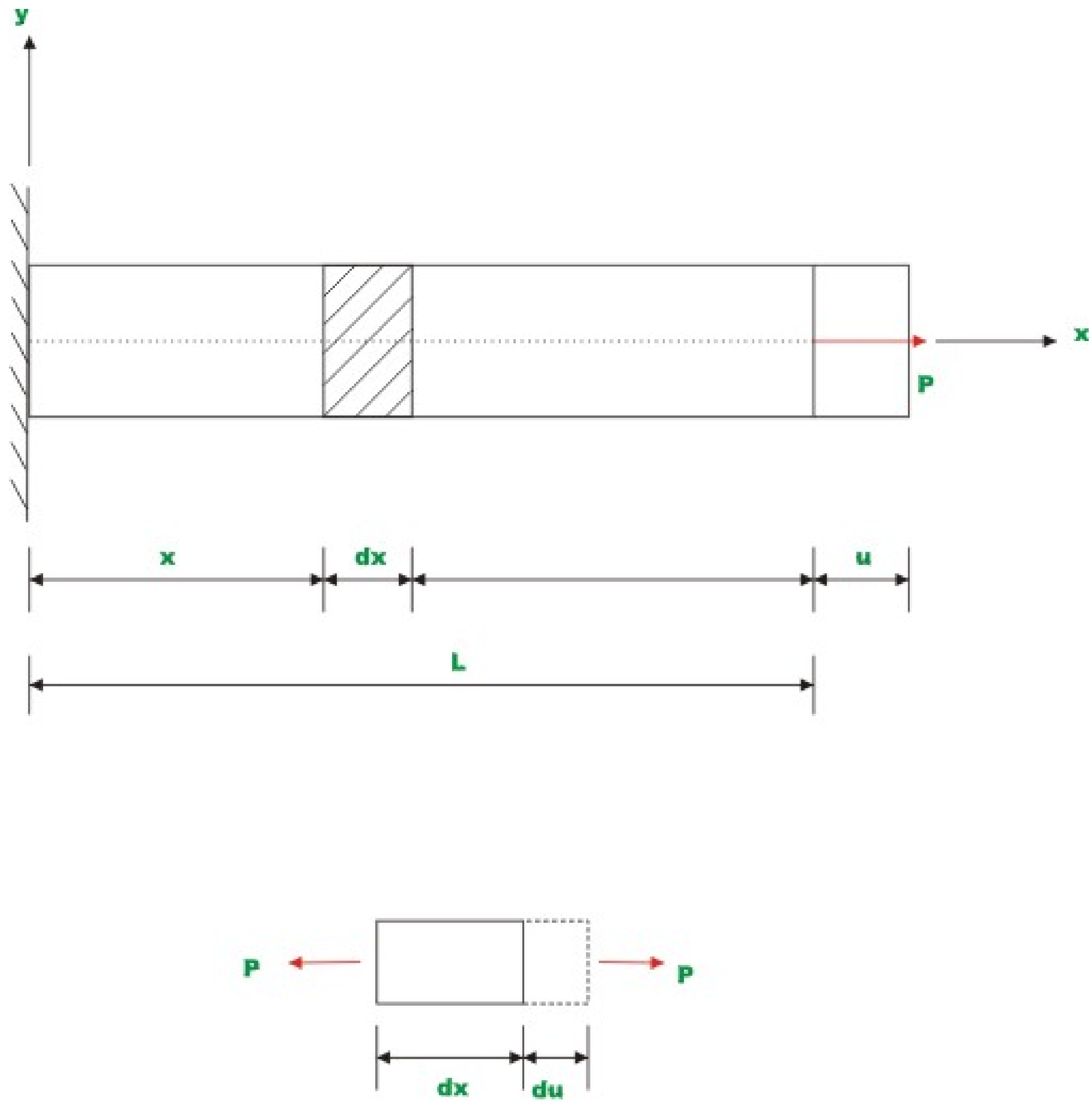

Fig 2.8

Now the work done by external loads $W = \frac{1}{2} P u$ (2.13)

In a conservative system, the external work is stored as the internal strain energy. Hence, the strain energy stored in the bar in axial deformation is,

$$U = \frac{1}{2} P u \tag{2.14}$$

Substituting equation (2.12) in (2.14) we get,

$$U = \int_0^L \frac{P^2}{2AE} dx \qquad (2.15)$$

2.3.2 Strain energy due to bending

Consider a prismatic beam subjected to loads as shown in the Fig. 2.9. The loads are assumed to act on the beam in a plane containing the axis of symmetry of the cross section and the beam axis. It is assumed that the transverse cross sections (such as AB and CD), which are perpendicular to centroidal axis, remain plane and perpendicular to the centroidal axis of beam (as shown in Fig 2.9).

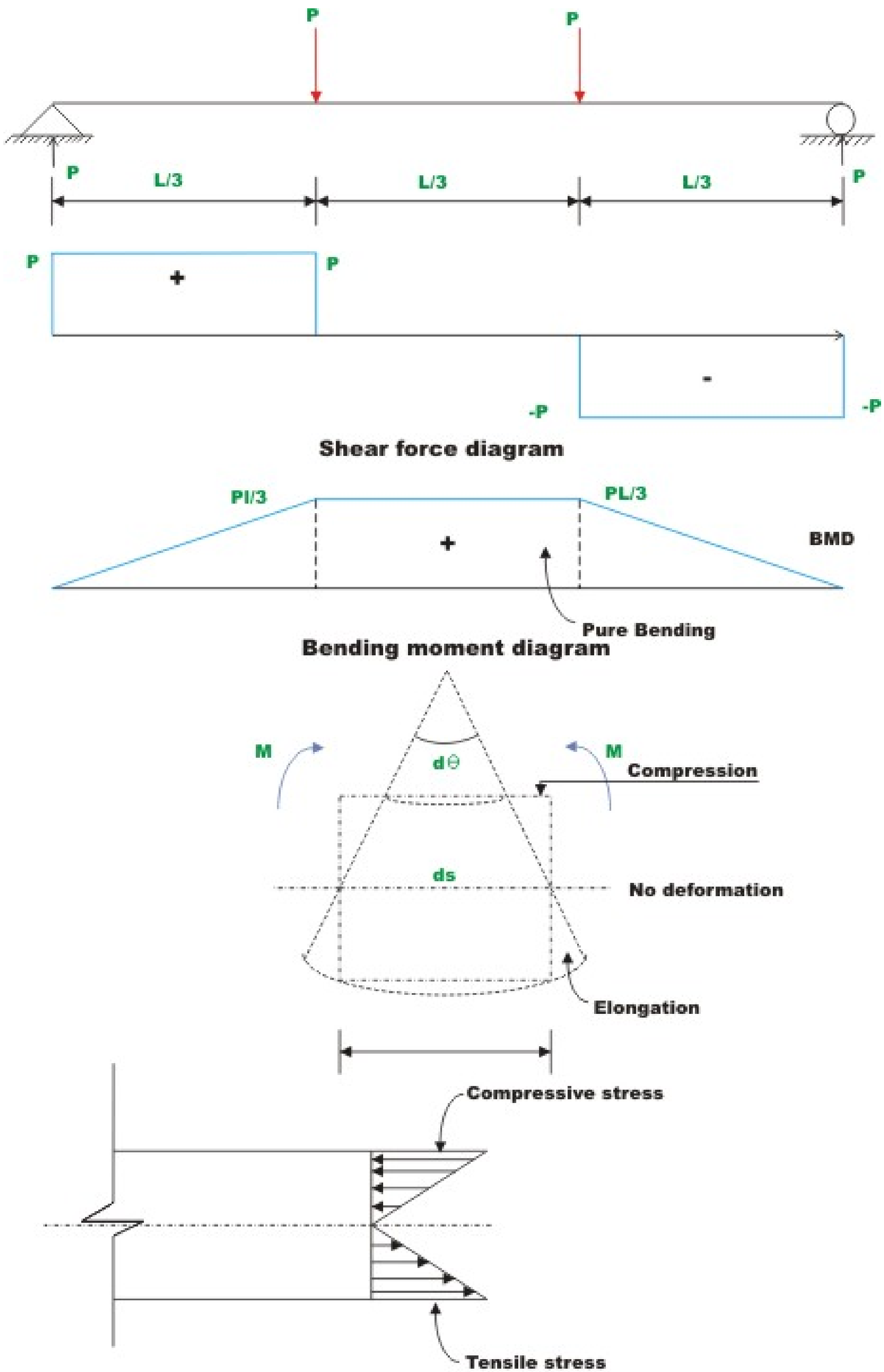

Fig. 2.9 BENDING DEFORMATION

Consider a small segment of beam of length ds subjected to bending moment as shown in the Fig. 2.9. Now one cross section rotates about another cross section by a small amount $d\theta$. From the figure,

$$d\theta = \frac{1}{R} ds = \frac{M}{EI} ds \tag{2.16}$$

where R is the radius of curvature of the bent beam and EI is the flexural rigidity of the beam. Now the work done by the moment M while rotating through angle $d\theta$ will be stored in the segment of beam as strain energy dU. Hence,

$$dU = \frac{1}{2} M \ d\theta \tag{2.17}$$

Substituting for $d\theta$ in equation (2.17), we get,

$$dU = \frac{1}{2} \frac{M^2}{EI} ds \tag{2.18}$$

Now, the energy stored in the complete beam of span L may be obtained by integrating equation (2.18). Thus,

$$U = \int_0^L \frac{M^2}{2EI} ds \tag{2.19}$$

2.3.3 Strain energy due to transverse shear

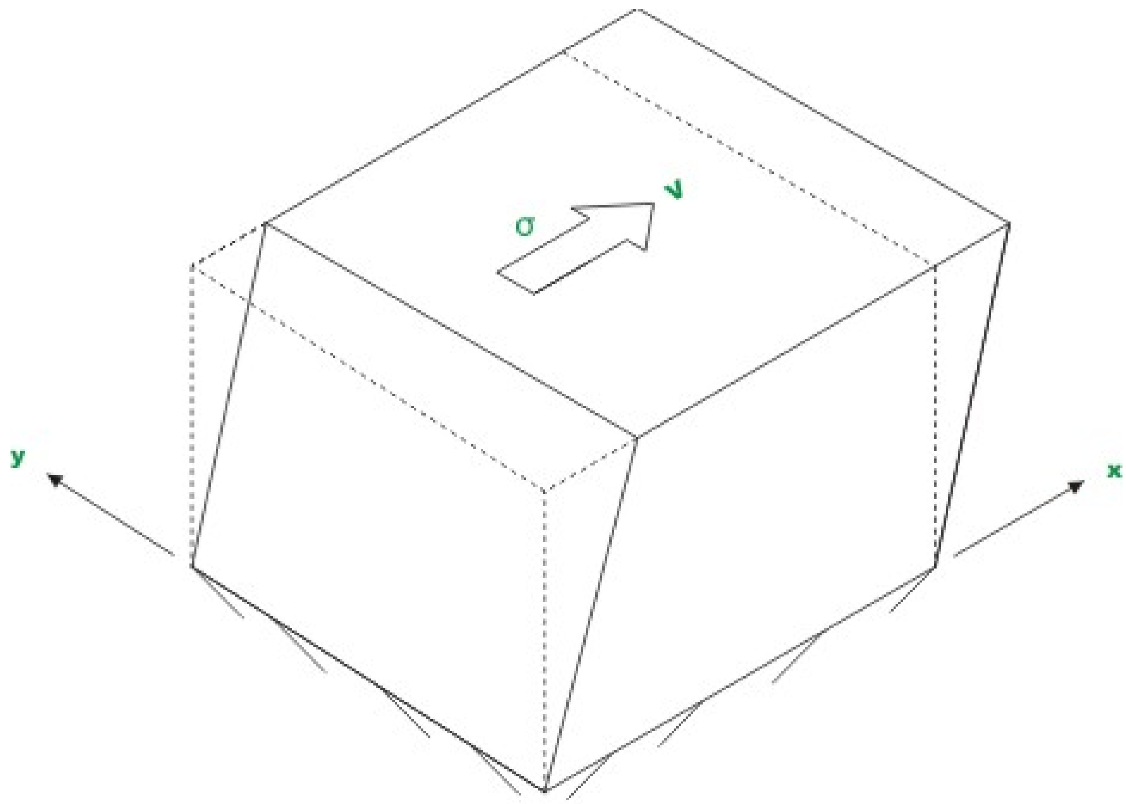

Fig. 2.10 (a) Shear Deformation

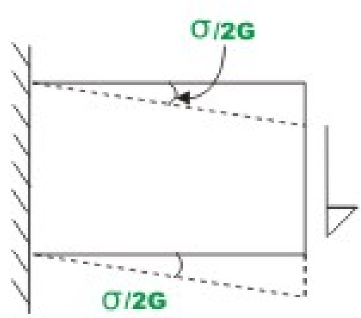

Fig. 2.10 (b)

The shearing stress on a cross section of beam of rectangular cross section may be found out by the relation

$$\tau = \frac{VQ}{bI_{ZZ}} \qquad (2.20)$$

where Q is the first moment of the portion of the cross-sectional area above the point where shear stress is required about neutral axis, V is the transverse shear force, b is the width of the rectangular cross-section and I_{zz} is the moment of inertia of the cross-sectional area about the neutral axis. Due to shear stress, the angle between the lines which are originally at right angle will change. The shear stress varies across the height in a parabolic manner in the case of a rectangular cross-section. Also, the shear stress distribution is different for different shape of the cross section. However, to simplify the computation shear stress is assumed to be uniform (which is strictly not correct) across the cross section. Consider a segment of length ds subjected to shear stress τ. The shear stress across the cross section may be taken as

$$\tau = k\frac{V}{A}$$

in which A is area of the cross-section and k is the form factor which is dependent on the shape of the cross section. One could write, the deformation du as

$$du = \Delta\gamma \; ds \qquad (2.21)$$

where $\Delta\gamma$ is the shear strain and is given by

$$\Delta\gamma = \frac{\tau}{G} = k\frac{V}{AG} \qquad (2.22)$$

Hence, the total deformation of the beam due to the action of shear force is

$$u = \int_0^L k\frac{V}{AG}ds \qquad (2.23)$$

Now the strain energy stored in the beam due to the action of transverse shear force is given by,

$$U = \frac{1}{2}Vu = \int_0^L \frac{kV^2}{2AG}ds \qquad (2.24)$$

The strain energy due to transverse shear stress is very low compared to strain energy due to bending and hence is usually neglected. Thus the error induced in assuming a uniform shear stress across the cross section is very small.

2.3.4 Strain energy due to torsion

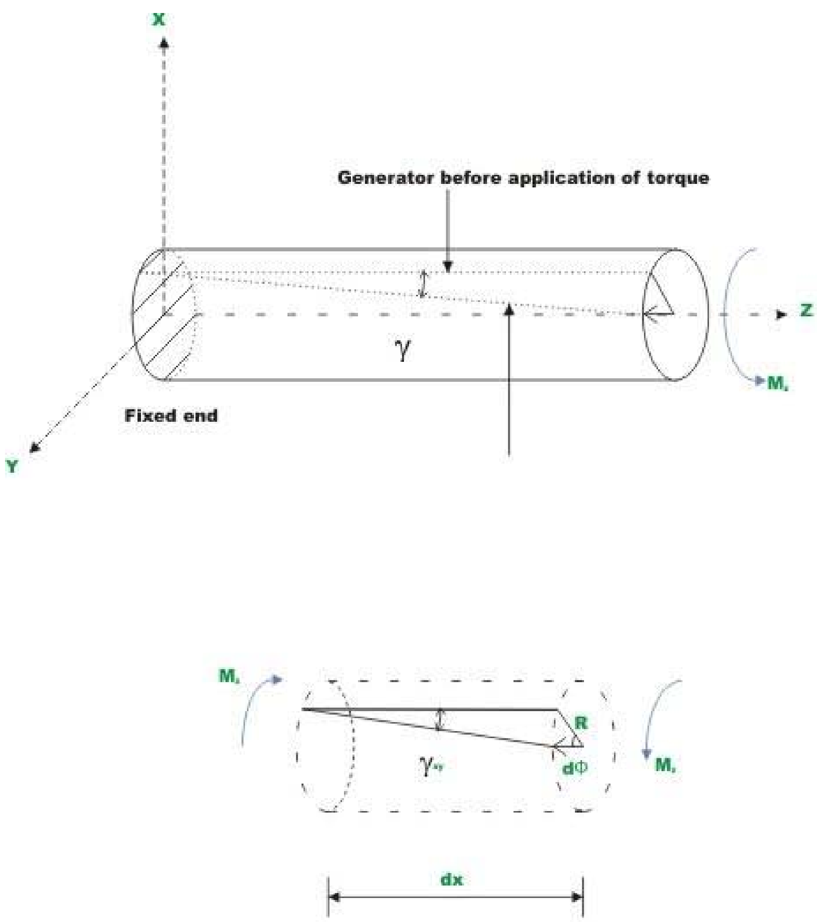

Fig 2.11 Generator after application of torque

Consider a circular shaft of length L radius R, subjected to a torque T at one end (see Fig. 2.11). Under the action of torque one end of the shaft rotates with respect to the fixed end by an angle $d\phi$. Hence the strain energy stored in the shaft is,

$$U = \frac{1}{2}T\phi \tag{2.25}$$

Consider an elemental length ds of the shaft. Let the one end rotates by a small amount $d\phi$ with respect to another end. Now the strain energy stored in the elemental length is,

$$dU = \frac{1}{2} T d\phi \tag{2.26}$$

We know that

$$d\phi = \frac{Tds}{GJ} \tag{2.27}$$

where, G is the shear modulus of the shaft material and J is the polar moment of area. Substituting for $d\phi$ from (2.27) in equation (2.26), we obtain

$$dU = \frac{T^2}{2GJ} ds \tag{2.28}$$

Now, the total strain energy stored in the beam may be obtained by integrating the above equation.

$$U = \int_0^L \frac{T^2}{2GJ} ds \tag{2.29}$$

Hence the elastic strain energy stored in a member of length s (it may be curved or straight) due to axial force, bending moment, shear force and torsion is summarized below.

1. Due to axial force $U_1 = \int_0^s \frac{P^2}{2AE} ds$

2. Due to bending $U_2 = \int_0^s \frac{M^2}{2EI} ds$

3. Due to shear $U_3 = \int_0^s \frac{V^2}{2AG} ds$

4. Due to torsion $U_4 = \int_0^s \frac{T^2}{2GJ} ds$

Summary

In this lesson, the principle of superposition has been stated and proved. Also, its limitations have been discussed. In section 2.3, it has been shown that the elastic strain energy stored in a structure is equal to the work done by applied loads in deforming the structure. The strain energy expression is also expressed for a 3-dimensional homogeneous and isotropic material in terms of internal stresses and strains in a body. In this lesson, the difference between elastic and inelastic strain energy is explained. Complementary strain energy is discussed. In the end, expressions are derived for calculating strain stored in a simple beam due to axial load, bending moment, transverse shear force and torsion.