Module – 5

Wave Propagation
Wave Propagation:

- Excitation of a compliant medium is not instantly felt at other points within the medium

- It takes time for the effects of the excitation to be felt at distant points

- The effects are felt in the form of waves that travel through the medium

- The manner in which these waves travel will control the effects they produce
Waves – Basics

Excitation  Direction of travel

Wave length $\lambda_c$
Waves – Basics

Particle motion
Seismic Waves:

When an earthquake occurs, different types of seismic waves are produced:

- **Body waves**
  - P-waves
  - S-waves
    - SV waves
    - SH waves

- **Surface waves**
  - Rayleigh waves
  - Love waves
Waves in Unbounded Media

One-Dimensional Wave Propagation

Longitudinal  Torsional  Flexural

Reference: Kramer (1996)
Longitudinal Waves in an Infinite Rod

\[ \sigma_{xo} = \sigma_x(x_0, t) \]

\[ \sigma_{xo} + \frac{\partial \sigma_{xo}}{\partial x} dx \]

\[ u = u(x_0, t) \]

\[ u + \frac{\partial u}{\partial x} dx \]

Reference: Kramer (1996)
Longitudinal Waves in an Infinite Rod

\[ \sigma_{xo} = \sigma_x(x_0, t) \]

\[ \sigma_{xo} + \frac{\partial \sigma_{xo}}{\partial x} dx \]

\[ v_p = \sqrt{\frac{M}{\rho}} \quad \text{p-wave velocity} \]

\[ \sigma_x = M \varepsilon_x, \quad \varepsilon_x = \frac{\partial u}{\partial x} \]

\[ \dot{u} = \frac{\partial u}{\partial t} = \varepsilon_x \frac{\partial x}{\partial t} = \frac{\sigma_x}{M} \frac{v_p}{\partial t} = \frac{\sigma_x}{M} \frac{v_p}{\partial t} = \frac{\sigma_x}{\rho v_p^2} \frac{v_p}{\partial t} = \frac{\sigma_x}{\rho v_p} \]

\[ \rho v_p \quad \text{specific impedance} \]

Reference: Kramer (1996)
Wave equation reduces to

\[ -\bar{\omega}^2 A \cos \bar{\omega}t - kx = -v^2 k^2 A \cos \bar{\omega}t - kx \]

Using complex notation the equivalent form of solution:

\[ u(x, t) = Ce^{i \bar{\omega}t - kx} + De^{i \bar{\omega}t + kx} \]

Reference: Kramer (1996)
Solution of the 1-D Equation of Motion

\[ \lambda = \frac{2\pi}{k} \quad T = \frac{2\pi}{\omega} \]

Motion is harmonic in time

Motion is harmonic in space

Reference: Kramer (1996)
Boundary Effects

At centerline, displacement is always zero
Stress doubles momentarily as waves pass each other
Boundary Effects (Fixed End)

\[ \sigma_0 \]

\[ u = 0 \]

Response at boundary is exactly the same as for case of two waves of same polarity traveling toward each other.

At fixed end, displacement is zero and stress is momentarily doubled. Polarity of reflected wave is same as that of incident wave.

Reference: Kramer (1996)
Boundary Effects

At centerline, stress is always zero
Particle velocity doubles momentarily as waves pass each other
Boundary Effects (Free End)

Response at boundary is exactly the same as for case of two waves of opposite polarity traveling toward each other.

At free end, stress is zero and displacement is momentarily doubled. Polarity of reflected wave is opposite that of incident wave.
Boundary Effects (Material Boundaries)

At material boundary, displacements must be continuous

\[ A_i + A_r = A_t \]

equilibrium must be satisfied

\[ \sigma_i + \sigma_r = \sigma_t \]

Reference : Kramer (1996)
Waves in a Layered Body

One-dimensional case: material boundary in an infinite rod

One-dimensional wave propagation at material interface. Incident and reflected waves travel in opposite directions in material 1. The transmitted wave travels through material 2 in the same direction as the incident wave.

*Reference: Kramer (1996)*
Waves in layered body (contd.)

The incident wave can be described by,

\[ \sigma_I(x, t) = \sigma_i e^{i \omega t - k_1 x} \]

The transmitted and the reflected wave can be described by

\[ \sigma_T(x, t) = \sigma_t e^{i \omega t - k_2 x} \]
\[ \sigma_R(x, t) = \sigma_r e^{i \omega t - k_1 x} \]

Assuming that the displacements associated with each of these waves are of the same harmonic form as the stresses,

\[ u_I(x, t) = A_i e^{i \omega t - k_1 x} \]
\[ u_R(x, t) = A_r e^{i \omega t - k_1 x} \]
\[ u_T(x, t) = A_t e^{i \omega t - k_2 x} \]

Reference: Kramer (1996)
Stress-strain and strain-displacement relationships can be used to relate the stress amplitudes to the displacement amplitudes

\[
\sigma_I(x,t) = M_1 \frac{\partial u_I(x,t)}{\partial x} = -ik_1 M_1 A_i e^{i \omega t - k_1 x}
\]

\[
\sigma_R(x,t) = M_1 \frac{\partial u_R(x,t)}{\partial x} = +ik_1 M_1 A_r e^{i \omega t - k_1 x}
\]

\[
\sigma_T(x,t) = M_2 \frac{\partial u_T(x,t)}{\partial x} = -ik_2 M_2 A_t e^{i \omega t - k_2 x}
\]

From these the stress amplitude are related to the displacement amplitudes by

\[
\sigma_i = -ik_1 M_1 A_i
\]

\[
\sigma_r = +ik_1 M_1 A_r
\]

\[
\sigma_t = -ik_2 M_2 A_t
\]

At the interface, both compatibility of displacements and continuity of stresses must be satisfied. The former requires that

\[
u_I(0,t) + u_R(0,t) = u_T(0,t)
\]

And the later one,

\[
\sigma_I(0,t) + \sigma_R(0,t) = \sigma_T(0,t)
\]

Reference: Kramer (1996)