

## Module 5 : Force Method - Introduction and applications

### Lecture 3 : Analysis of Statically Indeterminate Structure

#### Objectives

In this course you will learn the following

- Energy method for analysis of statically indeterminate structures.
- Illustrative examples for analysis of statically indeterminate structures using every method.

#### 5.4 Analysis of Statically Indeterminate Structures by Energy Method

Let a statically indeterminate structure has degree of indeterminacy as  $n$ . On the selected basic determinate structure apply the unknown forces  $R_1, R_2, \dots$  and  $R_n$ . Using the Eq. (4.16) the displacement  $\Delta_j$  in the direction of  $R_j$  is expressed by

$$\Delta_j = \frac{\partial U}{\partial R_j} \quad (j = 1, 2, \dots, n) \quad (5.1)$$

The equations (5.1) will provide the  $n$  linear simultaneous equations with  $n$  unknowns  $R_1, R_2, \dots$  and  $R_n$ . Since the  $\Delta_j$  is known, therefore, the solution of simultaneous equations will provide the desired  $R_j$  ( $j = 1, 2, \dots, n$ ).

For structures with members subjected to the axial forces only (i.e. pin-jointed structures), the equation (5.1) is re-written as

$$\Delta_j = \frac{\partial}{\partial R_j} \sum \left( \frac{P^2 L}{2AE} \right) = \sum \left( \frac{P \frac{\partial P}{\partial R_j} L}{AE} \right) \quad (5.2)$$

where  $P$  is the force in the member due to applied loading and unknown  $R_j$  ( $j = 1, 2, \dots, n$ ); and  $L$  and  $AE$  are length and axial rigidity of the member, respectively.

For structures with members subjected to the bending moments (i.e. beams and rigid-jointed frames), the equation (5.1) is re-written as

$$\Delta_j = \frac{\partial}{\partial R_j} \left( \int \frac{M^2 dx}{2EI} \right) = \int \frac{M \frac{\partial M}{\partial R_j} dx}{EI} \quad (5.3)$$

where  $M$  is the bending moment due to applied loading and unknown  $R_j$  ( $j = 1, 2, \dots, n$ ) at a small element of length  $dx$ ; and  $EI$  is the flexural rigidity.

**Example 5.14** A beam is suspended by three springs as shown in Figure 5.17(a). The flexibility of the springs  $AD$ ,  $BE$  and  $CF$  are  $f_1$ ,  $f_2$  and  $f_3$  respectively. The beam carries a load  $W$  at the middle of  $DE$ . Determine the force in the spring  $BE$  assuming (i) the beam to be stiff in comparison to the springs and (ii) flexible with flexural rigidity  $EI$ .

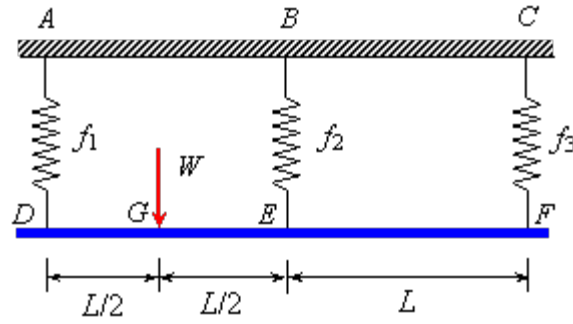


Figure 5.17(a)

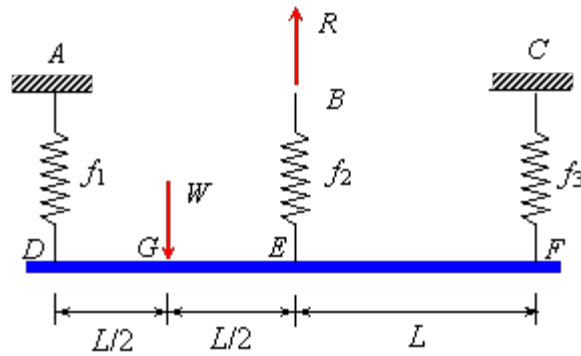


Figure 5.17(b)

**Solution:** The degree of static indeterminacy =  $3 - 2 = 1$ . Let the force in the spring  $BE$  be  $R$  as shown in Figure 5.17(b). Taking moment about point  $F$ , we have

$$F_{AD} \times 2L = W \times \frac{3L}{2} - RL$$

$$\therefore F_{AD} = \frac{3W}{4} - \frac{R}{2}$$

Similarly, taking moment about point  $D$ , we have

$$F_{CF} \times 2L = W \times \frac{L}{2} - RL$$

$$\therefore F_{CF} = \frac{W}{4} - \frac{R}{2}$$

**(i) When beam is rigid**

Total energy stored in the system is due to springs only as the beam is rigid. Thus,

$$U_s = \frac{1}{2} f_1 \left( \frac{3W}{4} - \frac{R}{2} \right)^2 + \frac{1}{2} f_2 R^2 + \frac{1}{2} f_3 \left( \frac{W}{4} - \frac{R}{2} \right)^2$$

Since the displacement of point  $E$  is zero in the vertical direction implying that

$$\frac{\partial U_s}{\partial R} = f_1 \left( \frac{3W}{4} - \frac{R}{2} \right) \left( -\frac{1}{2} \right) + f_2 R + f_3 \left( \frac{W}{4} - \frac{R}{2} \right) \left( -\frac{1}{2} \right)$$

or

$$0 = -\frac{W}{8}(3f_1 + f_3) + \frac{R}{4}(f_1 + 4f_2 + f_3)$$

$$\therefore R = \frac{W}{2} \left( \frac{3f_1 + f_3}{f_1 + 4f_2 + f_3} \right)$$

**(ii) When beam is flexible**

The total energy stored in the beam

$$U_b = U_{DG} + U_{GE} + U_{EF}$$

Span  $DG$  : (  $x$  measured from  $D$  )

$$M_x = \left( \frac{3W}{4} - \frac{R}{2} \right) x$$

$$\frac{\partial M_x}{\partial R} = -\frac{x}{2}$$

$$\frac{\partial U_{DG}}{\partial R} = \frac{1}{EI} \int_0^{L/2} \left[ \frac{3W}{4} - \frac{R}{2} \right] \left( -\frac{x^2}{2} \right) dx = \frac{1}{EI} \left[ \frac{3W}{4} - \frac{R}{2} \right] \left[ \frac{-L^3}{48} \right]$$

Span  $GE$  : (  $x$  measured from  $D$  )

$$M_x = \left( \frac{3W}{4} - \frac{R}{2} \right) x - W \left( x - \frac{L}{2} \right)$$

$$\frac{\partial M_x}{\partial R} = -\frac{x}{2}$$

$$\frac{\partial U_{GE}}{\partial R} = \frac{1}{EI} \int_{L/2}^L \left[ \left( \frac{3W}{4} - \frac{R}{2} \right) \left( -\frac{x^2}{2} \right) + W \left( x - \frac{L}{2} \right) \left( \frac{x}{2} \right) \right] dx$$

$$\frac{\partial U_{GE}}{\partial R} = \frac{1}{EI} \left[ \left( \frac{3W}{4} - \frac{R}{2} \right) \left( \frac{-L^3}{6} + \frac{L^3}{48} \right) + W \left( \frac{L^3}{6} - \frac{L^3}{8} - \frac{L^3}{48} + \frac{L^3}{32} \right) \right] = \frac{1}{EI} \left[ \left( \frac{3W}{4} - \frac{R}{2} \right) \left( \frac{-7L^3}{48} \right) + W \left( \frac{5L^3}{96} \right) \right]$$

Span  $EF$  : (  $x$  measured from  $F$  )

$$M_x = \left( \frac{W}{4} - \frac{R}{2} \right) x$$

$$\frac{\partial M_x}{\partial R} = -\frac{x}{2}$$

$$\frac{\partial U_{EF}}{\partial R} = \frac{1}{EI} \int_0^L \left[ \frac{W}{4} - \frac{R}{2} \right] \left( -\frac{x^2}{2} \right) dx = \frac{1}{EI} \left[ \frac{W}{4} - \frac{R}{2} \right] \left[ \frac{-L^3}{6} \right]$$

Thus,

$$\begin{aligned} \frac{\partial U_b}{\partial R} &= \frac{\partial U_{DG}}{\partial R} + \frac{\partial U_{GE}}{\partial R} + \frac{\partial U_{EF}}{\partial R} \\ &= \frac{1}{EI} \left[ \frac{3W}{4} - \frac{R}{2} \right] \left[ \frac{-L^3}{48} \right] + \frac{1}{EI} \left[ \left( \frac{3W}{4} - \frac{R}{2} \right) \left( \frac{-7L^3}{48} \right) + W \left( \frac{5L^3}{96} \right) \right] + \frac{1}{EI} \left[ \frac{W}{4} - \frac{R}{2} \right] \left[ \frac{-L^3}{6} \right] \end{aligned}$$

$$= \frac{1}{EI} \left[ -\frac{11WL^3}{96} + \frac{RL^3}{6} \right]$$

Total strain energy in the system

$$U = U_s + U_b$$

Hence, 
$$\frac{\partial U}{\partial R} = \frac{\partial U_s}{\partial R} + \frac{\partial U_b}{\partial R}$$

$$= -\frac{W}{8}(3f_1 + f_3) + \frac{R}{4}(f_1 + 4f_2 + f_3) + \frac{1}{EI} \left[ -\frac{11WL^3}{96} + \frac{RL^3}{6} \right]$$

Since  $\frac{\partial U}{\partial R} = 0 \Rightarrow -\frac{W}{8}(3f_1 + f_3) + \frac{R}{4}(f_1 + 4f_2 + f_3) + \frac{1}{EI} \left[ -\frac{11WL^3}{96} + \frac{RL^3}{6} \right] = 0$

$$R \left( \frac{L^3}{6EI} + \frac{f_1}{4} + f_2 + \frac{f_3}{4} \right) = W \left( \frac{11L^3}{96EI} + \frac{3}{8}f_1 + \frac{f_3}{8} \right)$$

$$\therefore R = \frac{W \left( \frac{11L^3}{96EI} + \frac{3}{8}f_1 + \frac{f_3}{8} \right)}{\left( \frac{L^3}{6EI} + \frac{f_1}{4} + f_2 + \frac{f_3}{4} \right)}$$

**Example 5.15** The free ends of two cantilever beams each of length  $L$  and flexural rigidity  $EI$  are joined together with a spring as shown in Figure 5.18(a). The stiffness of the spring is  $EI/L^3$ . Determine the force in the spring due to a concentrated load  $W$  acting at center of the lower cantilever.

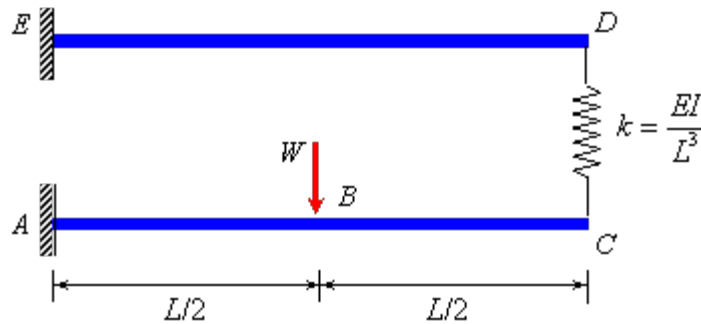


Figure 5.18(a)

**Solution:** Let the force in the spring be  $R$  as shown in Figure 5.18(b). According to the Castigliano's theorem

$$\frac{\partial U}{\partial R} = 0$$

where  $U$  is the total strain energy stored in the system.

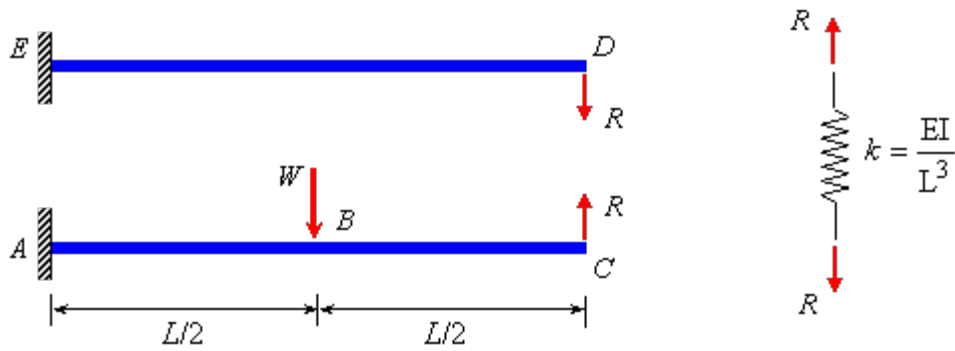


Figure 5.18(b)

Consider beam  $DE$  : (  $x$  measured from  $D$  ),

$$M_x = -Rx \text{ and } \frac{\partial M_x}{\partial R} = -x$$

$$\frac{\partial U_{DE}}{\partial R} = \frac{1}{EI} \int_0^L M_x \frac{\partial M_x}{\partial R} dx = \frac{RL^3}{3EI}$$

Consider the spring,

$$\frac{\partial U_s}{\partial R} = \frac{\partial}{\partial R} \left( \frac{R^2}{2k} \right) = \frac{RL^3}{EI}$$

Consider the beam  $AC$  ,

$$M_x = \begin{cases} Rx & 0 < x < L/2 \text{ for part } CB \\ R(x+L/2) - Wx & 0 < x < L/2 \text{ for part } AB \end{cases}$$

$$\frac{\partial U_{BC}}{\partial R} = \frac{1}{EI} \int_0^{L/2} (Rx) x dx = \frac{RL^3}{24EI}$$

$$\frac{\partial U_{AB}}{\partial R} = \frac{1}{EI} \int_0^{L/2} [R(x+L/2) - Wx] (x+L/2) dx = \frac{7RL^3}{24EI} - \frac{5WL^3}{48EI}$$

$$\text{Since } \frac{\partial U}{\partial R} = 0 \Rightarrow \frac{RL^3}{3EI} + \frac{RL^3}{EI} + \frac{RL^3}{24EI} + \frac{7RL^3}{24EI} - \frac{5WL^3}{48EI} = 0$$

$$\therefore R = \frac{W}{16}$$

**Example 5.16** Find the expression for the prop reaction in the propped cantilever beam shown in the Figure 5.19(a).

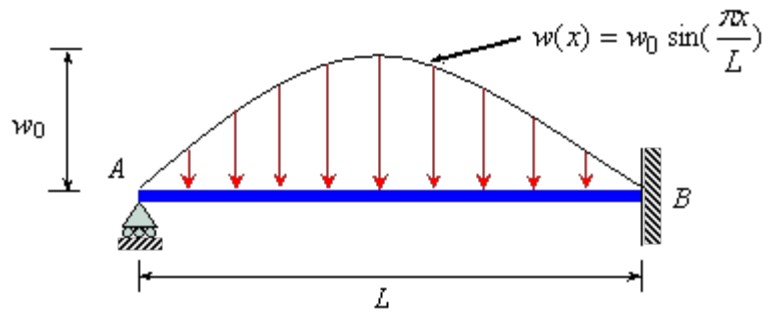


Figure 5.19(a)

**Solution:** Let reaction at support A be  $R$ . According to the Castigliano's theorem

$$\frac{\partial U}{\partial R} = 0$$

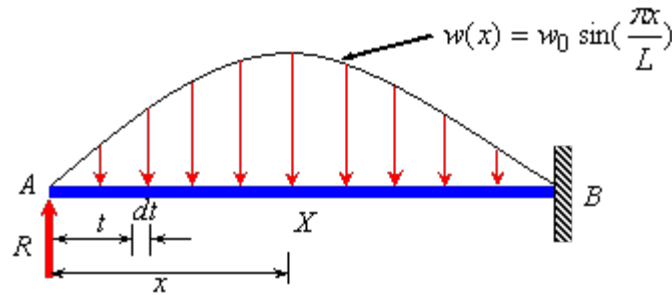


Figure 5.19(b)

The bending moment at any point  $X$  at a distance  $x$  from  $A$  is given by

$$\begin{aligned} M_x &= Rx - \int_0^x w(t) dt (x-t) \\ &= Rx - \int_0^x w_0 \sin\left(\frac{\pi t}{L}\right) (x-t) dt \\ &= Rx + \frac{w_0}{\pi^2} \sin(\pi x) - \frac{w_0}{\pi} x \end{aligned}$$

$$\frac{\partial M_x}{\partial R} = x$$

$$\begin{aligned} \frac{\partial U}{\partial R} &= \frac{1}{EI} \int_0^L \left( Rx + \frac{w_0}{\pi^2} \sin(\pi x) - \frac{w_0}{\pi} x \right) x dx \\ &= \frac{RL^3}{3EI} - \frac{w_0 L^4}{3\pi EI} - \frac{w_0 L^4}{\pi^3 EI} \end{aligned}$$

$$\text{Since } \frac{\partial U}{\partial R} = 0 \quad \Rightarrow \quad \frac{RL^3}{3EI} - \frac{w_0 L^4}{3\pi EI} - \frac{w_0 L^4}{\pi^3 EI} = 0$$

$$\therefore R = \frac{(\pi^2 - 3)}{\pi^3} w_0 L$$

**Example 5.17** Determine the force in various members of the pin-jointed frame shown in Figure 5.20(a). Length and  $AE$  is constant for all members.

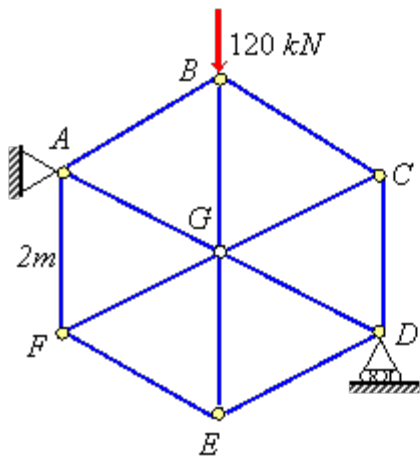


Figure 5.20(a)

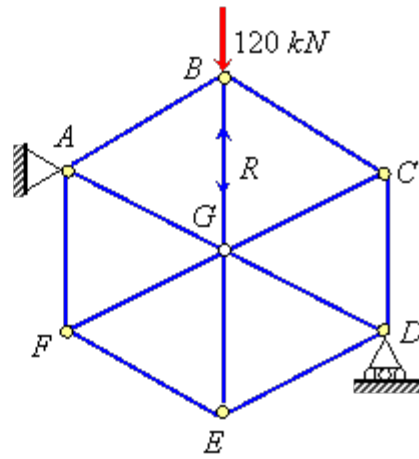


Figure 5.20(b)

**Solution:** The static indeterminacy of the pin-jointed frame is  $=12+3-7\times 2 = 1$ . Let the force in the member  $BG$  be  $R$  as shown in Figure 5.20(b). According to the Castigliano's theorem

$$\frac{\partial U}{\partial R} = 0$$

The computation of  $\partial U / \partial R$  is made in Table 5.6.

Table 5.6

Member	Length, $L$ ( m )	$P$	$\frac{\partial P}{\partial R}$	$P \frac{\partial P}{\partial R} L$	Final force (kN)
$AB$	2	$-120 + R$	1	$2 R - 240$	-40
$AG$	2	$120 - R$	-1	$2 R - 240$	40
$AF$	2	$-60 + R$	1	$2 R - 120$	20
$BC$	2	$-120 + R$	1	$2 R - 240$	-40
$BG$	2	$-R$	-1	$2 R$	-80
$CD$	2	$-120 + R$	1	$2 R - 240$	-40
$CG$	2	$120 - R$	-1	$2 R - 240$	40
$DE$	2	$-60 + R$	-1	$2 R - 120$	20
$DG$	2	$60 - R$	-1	$2 R - 120$	-20
$EF$	2	$-60 + R$	1	$2 R - 120$	20
$EG$	2	$60 - R$	-1	$2 R - 120$	-20
$FG$	2	$60 - R$	-1	$2 R - 120$	-20

$$\sum 24R - 1920$$

$$\frac{\partial U}{\partial R} = \frac{1}{AE} \left( P \frac{\partial P}{\partial R} L \right) = 0$$

or

$$\frac{1}{AE} (24R - 1920) = 0$$

$$\therefore R = 80 \text{ kN}$$

The final force in various members of the frame is shown in Table 5.6.

**Example 5.18** Determine the force in various members of the pin-jointed frame as shown in Figure 5.21(a), if the member  $BC$  is short by an amount of  $\Delta$ . All members of the frame have same axial rigidity as  $AE$ .

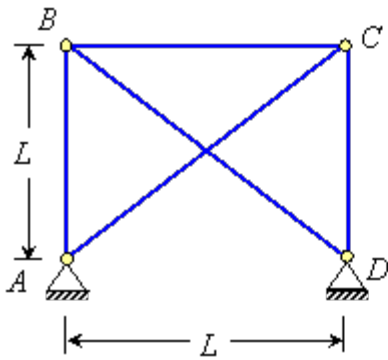


Figure 5.21(a)

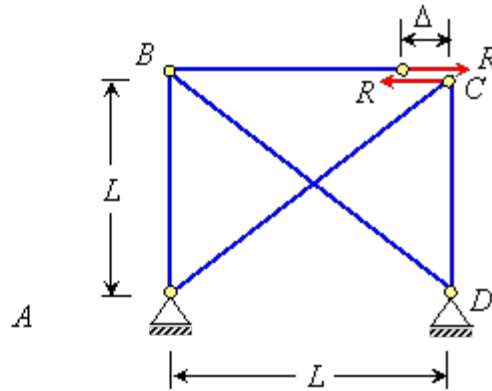


Figure 5.21(b)

**Solution:** The static indeterminacy of the pin-jointed frame is  $= 5 + 4 - 2 \times 4 = 1$ . Since the member  $BC$  is short by an amount of  $\Delta$ , therefore, apply a force  $R$  in the member  $BC$  such that displacement in the direction of  $R$  is  $\Delta$ . Thus, according to the Castigliano's theorem.

$$\frac{\partial U}{\partial R} = \Delta$$

The computation of  $\partial U / \partial R$  is made in Table 5.7.

Table 5.7

Member	Length, $L$ ( m )	$F$	$\frac{\partial F}{\partial R}$	$F \frac{\partial F}{\partial R} L$	Final force $\left( \frac{AE\Delta}{(3+4\sqrt{2})L} \right)$
$AB$	$L$	$R$	1	$RL$	1
$AC$	$\sqrt{2}L$	$-\sqrt{2}R$	$-\sqrt{2}$	$2\sqrt{2}RL$	$-\sqrt{2}$
$BC$	$L$	$R$	1	$RL$	1
$BD$	$L$	$-\sqrt{2}R$	$-\sqrt{2}$	$2\sqrt{2}RL$	$-\sqrt{2}$
$CD$	$\sqrt{2}L$	$R$	1	$RL$	1

$$\Sigma (3+4\sqrt{2})RL$$

$$\frac{\partial U}{\partial R} = \frac{1}{AE} \left( F \frac{\partial F}{\partial R} L \right) = \Delta$$

or

$$\frac{1}{AE} (3+4\sqrt{2})RL = \Delta$$

$$\therefore R = \frac{AE\Delta}{(3+4\sqrt{2})L}$$

The final force in various members of the frame is shown in Table 5.7.

**Example 5.19** Determine the horizontal reaction of the portal frame shown in Figure 5.22(a) by energy method. Also, calculate the horizontal reaction when the member  $BC$  is subjected to distributed load,  $w$  over entire length.



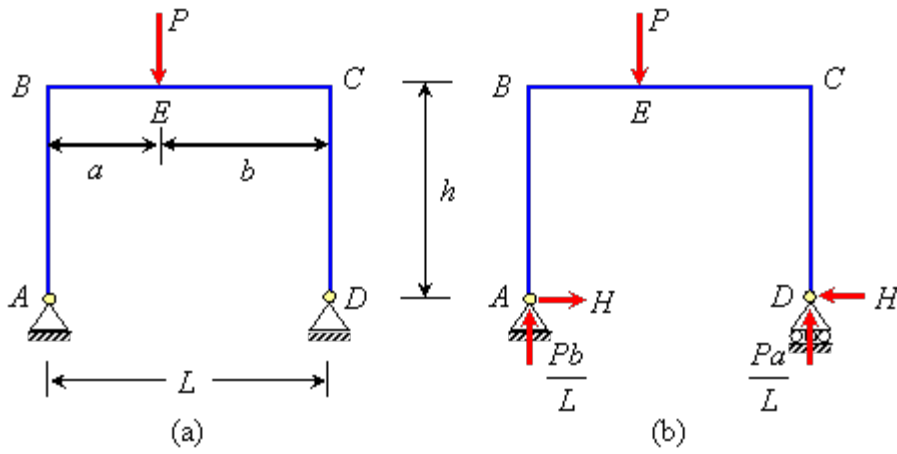


Figure 5.22(a)-(b)

**Solution:** Static indeterminacy of the frame = 1.

Let the horizontal reaction,  $H$  at  $D$  be the redundant. The reaction at  $A$  and  $D$  are

$$H_A = H; \quad H_D = H \quad R_A = Pb/L \quad \text{and} \quad R_D = Pa/L$$

For the span  $AB$  ( $x$  measured from  $A$ ),  $M_x = Hx$  and  $\frac{\partial M_x}{\partial H} = x$

$$\frac{\partial U_{AB}}{\partial H} = \frac{Hh^3}{3EI}$$

For the span  $BE$  ( $x$  measured from  $B$ ),  $M_x = Hh - \frac{Pb}{L}x$  and  $\frac{\partial M_x}{\partial H} = h$

$$\frac{\partial U_{BE}}{\partial H} = \frac{1}{EI} \int_0^a \left[ Hh - \frac{Pb}{L}x \right] h dx = \frac{Hh^2 a}{EI} - \frac{Pa^2 bh}{2EIL}$$

For the span  $CD$  ( $x$  measured from  $D$ ),  $M_x = Hx$  and  $\frac{\partial M_x}{\partial H} = x$

$$\frac{\partial U_{CD}}{\partial H} = \frac{Hh^3}{3EI}$$

For the span  $CE$  ( $x$  measured from  $C$ ),  $M_x = Hh - \frac{Pa}{L}x$  and  $\frac{\partial M_x}{\partial H} = h$

$$\frac{\partial U_{CE}}{\partial H} = \frac{Hh^2 b}{EI} - \frac{Pab^2 h}{2EIL}$$

$$\text{Since } \frac{\partial U}{\partial H} = 0 \quad \Rightarrow \quad \frac{Hh^3}{3EI} + \frac{Hh^2 a}{EI} - \frac{Pa^2 bh}{2EIL} + \frac{Hh^2 b}{EI} - \frac{Pab^2 h}{2EIL} + \frac{Hh^3}{3EI}$$

$$\frac{2Hh^2}{3} + HhL - \frac{Pab}{2} = 0$$

$$H = \frac{3Pab}{2h(2h + 3L)}$$

Horizontal reaction due to udl,  $w$  over BC :

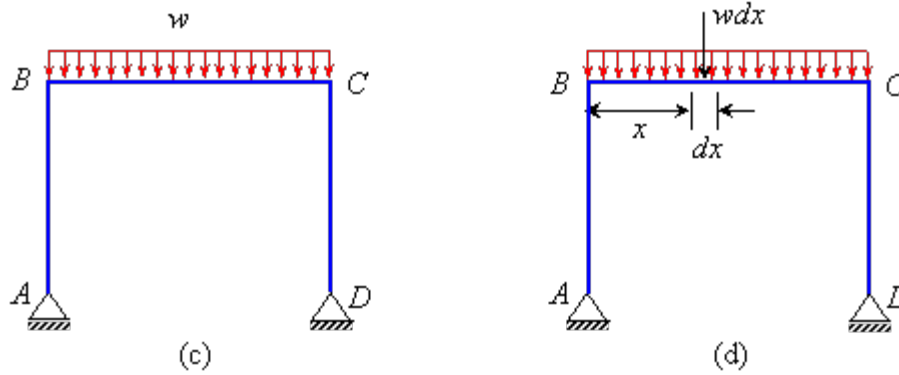


Figure 5.22(c)-(d)

The horizontal reaction due to small incremental load  $w dx$  is given by

$$dH = \frac{3w dx \cdot x(L-x)}{2h(2h+3L)}$$

(using the expression derived earlier for concentrated force and putting  $P = w dx$ ,  $a = x$  and  $b = L - x$ ).

The horizontal reaction due to entire distributed load

$$\begin{aligned} H &= \int dH \\ &= \int_0^L \frac{3w dx \cdot x(L-x)}{2h(2h+3L)} \\ &= \frac{3w}{2h(2h+3L)} \int_0^L x(L-x) dx \\ &= \frac{wL^3}{4h(2h+3L)} \end{aligned}$$

**Example 5.20** Analyze the portal frame shown in Figure 5.23 by strain energy method.

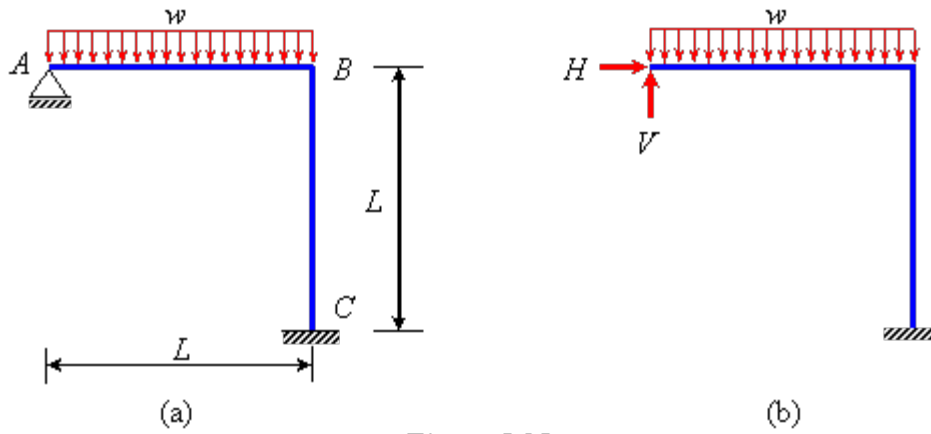


Figure 5.23

**Solution:** Static indeterminacy of the frame = 2. Horizontal and vertical reactions at A are taken as redundant.

For the span AB ( $x$  measured from A),  $M_x = Vx - wx^2/2$

$$\frac{\partial M_x}{\partial H} = 0 \quad \text{and} \quad \frac{\partial M_x}{\partial V} = x$$

$$\frac{\partial U_{AB}}{\partial H} = 0$$

$$\frac{\partial U_{AB}}{\partial V} = \frac{1}{EI} \int_0^L [Vx - wx^2/2][x] dx = \frac{VL^3}{3EI} - \frac{wL^4}{8EI}$$

$$M_x = Hx + VL - wL^2/2$$

$$\frac{\partial M_x}{\partial H} = x \quad \text{and} \quad \frac{\partial M_x}{\partial V} = L$$

$$\frac{\partial U_{BC}}{\partial H} = \frac{1}{EI} \int_0^L [Hx + VL - wL^2/2][x] dx = \frac{VL^3}{2EI} + \frac{HL^3}{3EI} - \frac{wL^4}{4EI}$$

$$\frac{\partial U_{BC}}{\partial V} = \frac{1}{EI} \int_0^L [Hx + VL - wL^2/2][L] dx = \frac{VL^3}{EI} + \frac{HL^3}{2EI} - \frac{wL^4}{2EI}$$

Since  $\frac{\partial U}{\partial H} = 0 \Rightarrow \frac{VL^3}{2EI} + \frac{HL^3}{3EI} - \frac{wL^4}{4EI} = 0$

or  $6V + 4H = 3wL$  (i)

and  $\frac{\partial U}{\partial V} = 0 \Rightarrow \frac{VL^3}{3EI} - \frac{wL^4}{8EI} + \frac{VL^3}{EI} + \frac{HL^3}{2EI} - \frac{wL^4}{2EI} = 0$

or  $32V + 12H = 15wL$  (ii)

Solving Eqs (i) and (ii) for  $H$  and  $V$ ,

$$H = \frac{3wL}{28} \quad \text{and} \quad V = \frac{3wL}{7}$$

**Example 5.21** Determine the support reactions of the continuous beam as shown in Figure 5.24(a) if the beam is assumed to be subjected to a linear temperature gradient such that the top surface of the beam is at temperature  $T_t$  and lower at  $T_b$ . The beam is uniform having flexural rigidity as  $EI$  and depth  $d$ . The coefficient of thermal expansion for beam material is  $\alpha$ .

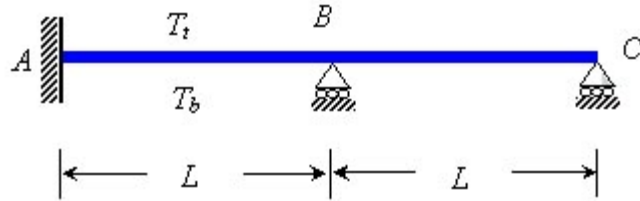


Figure 5.24(a)

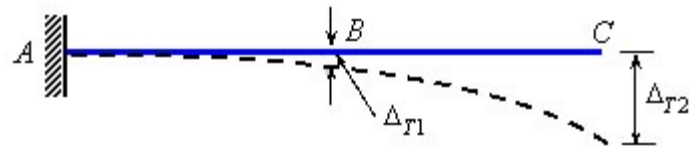


Figure 5.24(b)

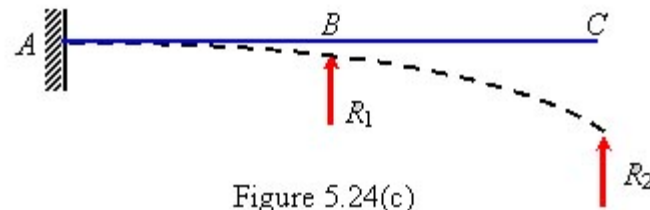


Figure 5.24(c)

**Solution:** The degree of static indeterminacy = 2. Remove the supports at  $B$  and  $C$  and allow the beam to deflect freely under the temperature variation. The deflection of the points  $B$  and  $C$  of the beam due to temperature variation

$$\Delta_{T1} = \frac{\alpha(T_t - T_b)L^2}{2d} \quad (i)$$

$$\Delta_{T2} = \frac{2\alpha(T_t - T_b)L^2}{d} \quad (ii)$$

Apply the forces  $R_1$  and  $R_2$  at point  $B$  and  $C$ , respectively. According to Castigliano's theorem

$$\frac{\partial U}{\partial R_1} = \Delta_{T1} = \frac{\alpha(T_t - T_b)L^2}{2d} \quad (iii)$$

$$\frac{\partial U}{\partial R_2} = \Delta_{T2} = \frac{2\alpha(T_t - T_b)L^2}{d} \quad (iv)$$

Consider  $BC$ : ( $x$  measured from  $C$ )

$$M_x = R_2 x, \quad \frac{\partial M_x}{\partial R_1} = 0, \quad \frac{\partial M_x}{\partial R_2} = x$$

$$\frac{\partial U_{BC}}{\partial R_1} = 0$$

$$\frac{\partial U_{BC}}{\partial R_2} = \frac{1}{EI} \int_0^L R_2 x \cdot x \cdot dx = \frac{R_2 L^3}{3EI}$$

Consider AB : ( x measured from B )

$$M_x = R_1 x + R_2 (L + x) \quad ; \quad \frac{\partial M_x}{\partial R_1} = x, \quad \frac{\partial M_x}{\partial R_2} = (L + x)$$

$$\frac{\partial U_{AB}}{\partial R_1} = \frac{1}{EI} \int_0^L \{ R_1 x + R_2 (L + x) \} x dx = \frac{R_1 L^3}{3EI} + \frac{5R_2 L^3}{6EI}$$

$$\frac{\partial U_{AB}}{\partial R_2} = \frac{1}{EI} \int_0^L \{ R_1 x + R_2 (L + x) \} (L + x) dx = \frac{5R_1 L^3}{6EI} + \frac{7R_2 L^3}{3EI}$$

Thus,

$$\frac{\partial U}{\partial R_1} = \frac{R_1 L^3}{3EI} + \frac{5R_2 L^3}{6EI} = \frac{\alpha(T_t - T_b)L^2}{2d}$$

or

$$2R_1 + 5R_2 = \frac{3EI\alpha(T_t - T_b)}{dL} \quad (v)$$

Similarly,

$$\frac{\partial U}{\partial R_2} = \frac{5R_1 L^3}{6EI} + \frac{7R_2 L^3}{3EI} + \frac{R_2 L^3}{3EI} = \frac{2\alpha(T_t - T_b)L^2}{d}$$

or

$$5R_1 + 16R_2 = \frac{12EI\alpha(T_t - T_b)}{dL} \quad (vi)$$

Solving eqs. (v) and (vi)

$$R_1 = -\frac{12EI\alpha(T_t - T_b)}{7dL} \quad \text{and} \quad R_2 = \frac{9EI\alpha(T_t - T_b)}{7dL}$$

The reactions of the beam are shown in Figure 5.24(d).

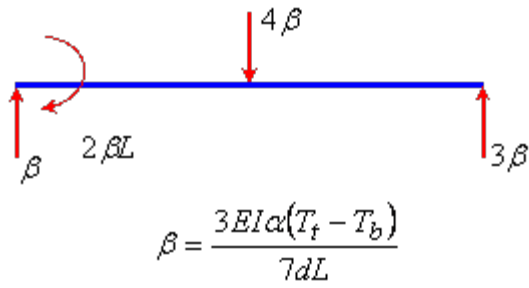


Figure 5.24(d) Reactions of the beam

### Recap

In this course you have learnt the following

- Energy method for analysis of statically indeterminate structures.
- Illustrative examples for analysis of statically indeterminate structures using every method.