Module 5 : Force Method - Introduction and applications

Lecture 2 : The Force Method

Objectives

In this course you will learn the following

- Concept of force method for analysis of statically indeterminate structure.
- Selection of the basic determinate structure.
- Illustration of force method by numerical examples.

5.3 The Force Method

The force method is used to calculate the response of statically indeterminate structures to loads and/or imposed deformations. The method is based on transforming a given structure into a statically determinate primary system and calculating the magnitude of statically redundant forces required to restore the geometric boundary conditions of the original structure. The force method (also called the flexibility method or method of consistent deformation) is used to calculate reactions and internal forces in statically indeterminate structures due to loads and imposed deformations.

The basic steps in the force method are as follows:

(a) Determine the degree of static indeterminacy, *n* of the structure.

(b) Transform the structure into a statically determinate system by releasing a number of static constraints equal to the degree of static indeterminacy, *n*. This is accomplished by releasing external support conditions or by creating internal hinges. The system thus formed is called the basic determinate structure .

(c) For a given released constraint *j*, introduce an unknown redundant force R_j corresponding to the type and direction of the released constraint.

(d) Apply the given loading or imposed deformation to the basic determinate structure . Use suitable method (given in Chapter 4) to calculate displacements at each of the released constraints in the basic determinate structure .

(e) Solve for redundant forces R_j (j =1 to n) by imposing the compatibility conditions of the original structure. These conditions transform the basic determinate structure back to the original structure by finding the combination of redundant forces that make displacement at each of the released constraints equal to zero.

It can thus be seen that the name force method was given to this method because its primary computational task is to calculate unknown forces , i.e. the redundant forces R_1 through R_n.

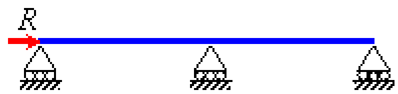
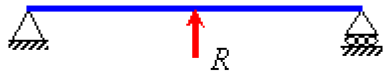
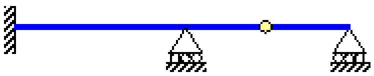
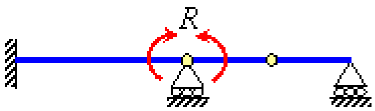
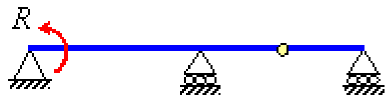
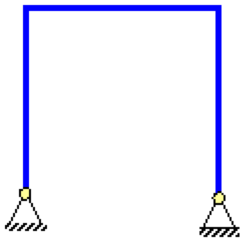
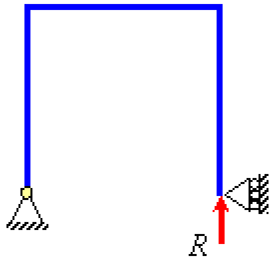
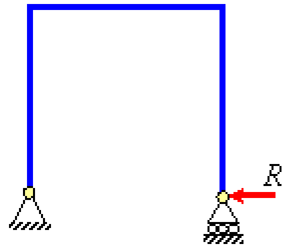
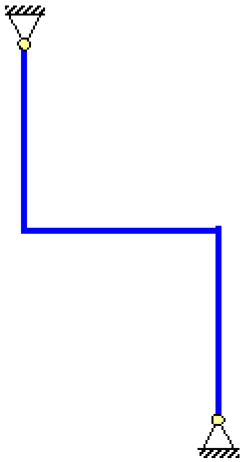
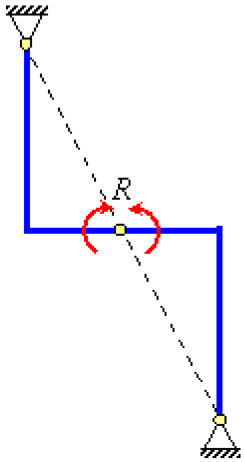
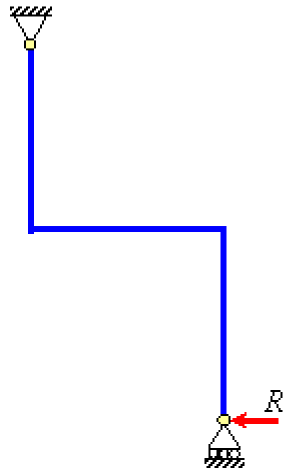
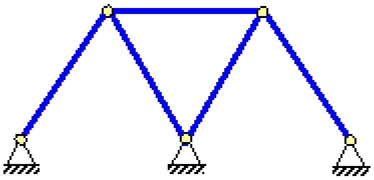
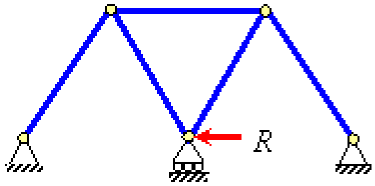
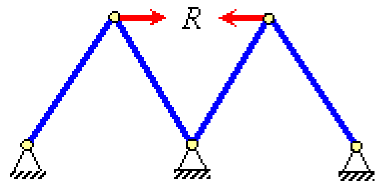
5.3.1 Selection of the basic determinate structure

There is no limit to the number of different basic determinate structure that can be generated for a given structure. The choice of structure, however, must ensure that the primary system is stable. In addition, it is recommended that the basic determinate structure be chosen to minimize computational effort and maximize computational accuracy.

(a) Stability of Basic determinate structure

It is not sufficient merely to release the correct number of statical constraints in generating a basic determinate structure. Care must be taken to ensure that the basic determinate structure is stable. This fact is explained in the Table 5.1 where any arbitrary release of constraint can result into the unstable basic determinate structure.

Table 5.1 Selection of basic determinate structure

Given indeterminate structure	Unstable basic determinate structure	Stable basic determinate structure
		
		
		
		
		

(b) Choice of Basic determinate Structure for minimum Computation work

The computational effort required in calculating the response of a given structure using the force method can vary significantly depending on the choice of basic determinate structure. In this regard, there are two issues to consider:

1. Select the basic determinate structure such as the displacements can be easily computed (i.e. converting it into simple structure).

2. Select the basic determinate structure to maximize the number of flexibility coefficients equal to zero.

These issues are illustrated in the following examples.

Consider a fixed beam as shown in Figure 5.8(a). The beam is non-prismatic with degree of indeterminacy 2. Three basic determinate structures are shown in Figures 5.8(b), (c) and (d). Among the three structures the computation effort will be minimum for the beam as in Figure 5.8(b) as the resulting basic determinate structure consists of two uniform cantilever beams.

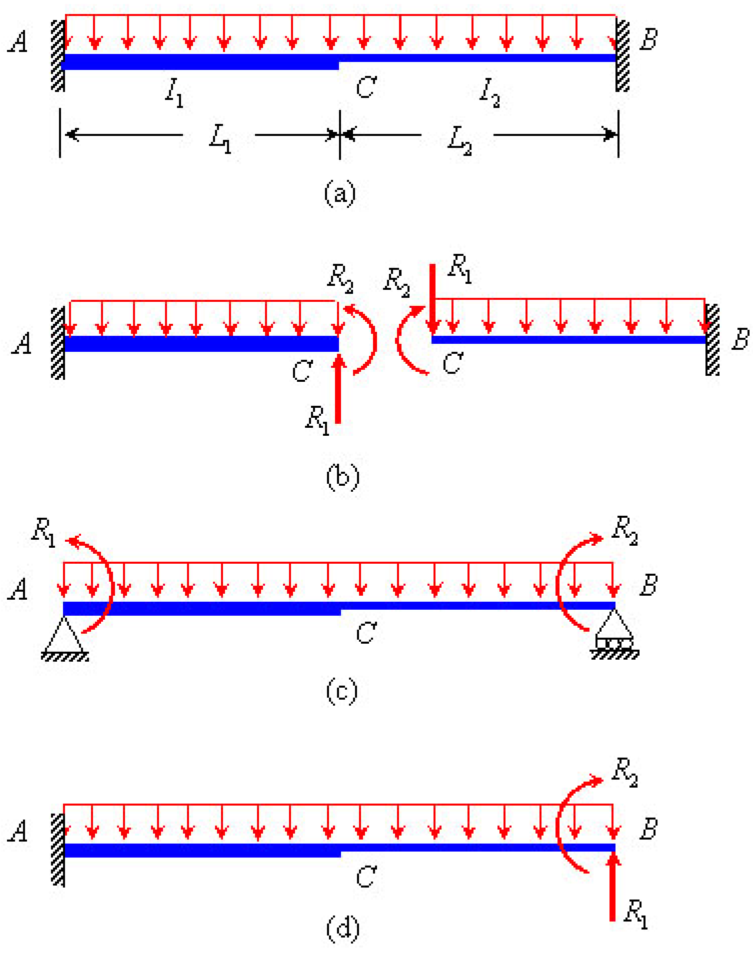


Figure 5.8(a)-(d)

The another structure under consideration is a four-span continuous beam as shown in Figure 5.9. The degree of static indeterminacy of the beam is 3. Two basic determinate structures are illustrated. On the left-hand side of the figure, the basic determinate structure is formed by releasing moment in the beam at the three interior supports. On the right-hand side of the figure, the basic determinate structure is formed by releasing the vertical reaction at the three interior supports.

For each basic determinate structure, bending moments M_p (bending moment in the basic determinate structure due to applied loading) M_1 (due to R_1), M_2 (due to R_2), and M_3 (due to R_3) are plotted. For the basic determinate structure on the left-hand side of the diagram, all integrations required for calculating the deflections can be easily performed using integration tables. In addition, for the left-hand side of the diagram, several deflection coefficients are zero. On the right-hand side, however, all coefficients are nonzero. The choice of basic determinate structure on the left allowed the influence of a given redundant force R_j to be restricted to a relatively small portion of the structure (two spans in this particular case). For the structure on the right-hand side, the influence of a given redundant force *R* j is felt throughout the structure. It can be concluded that the basic determinate structure on the left-hand side is preferable because it reduces the computational effort.

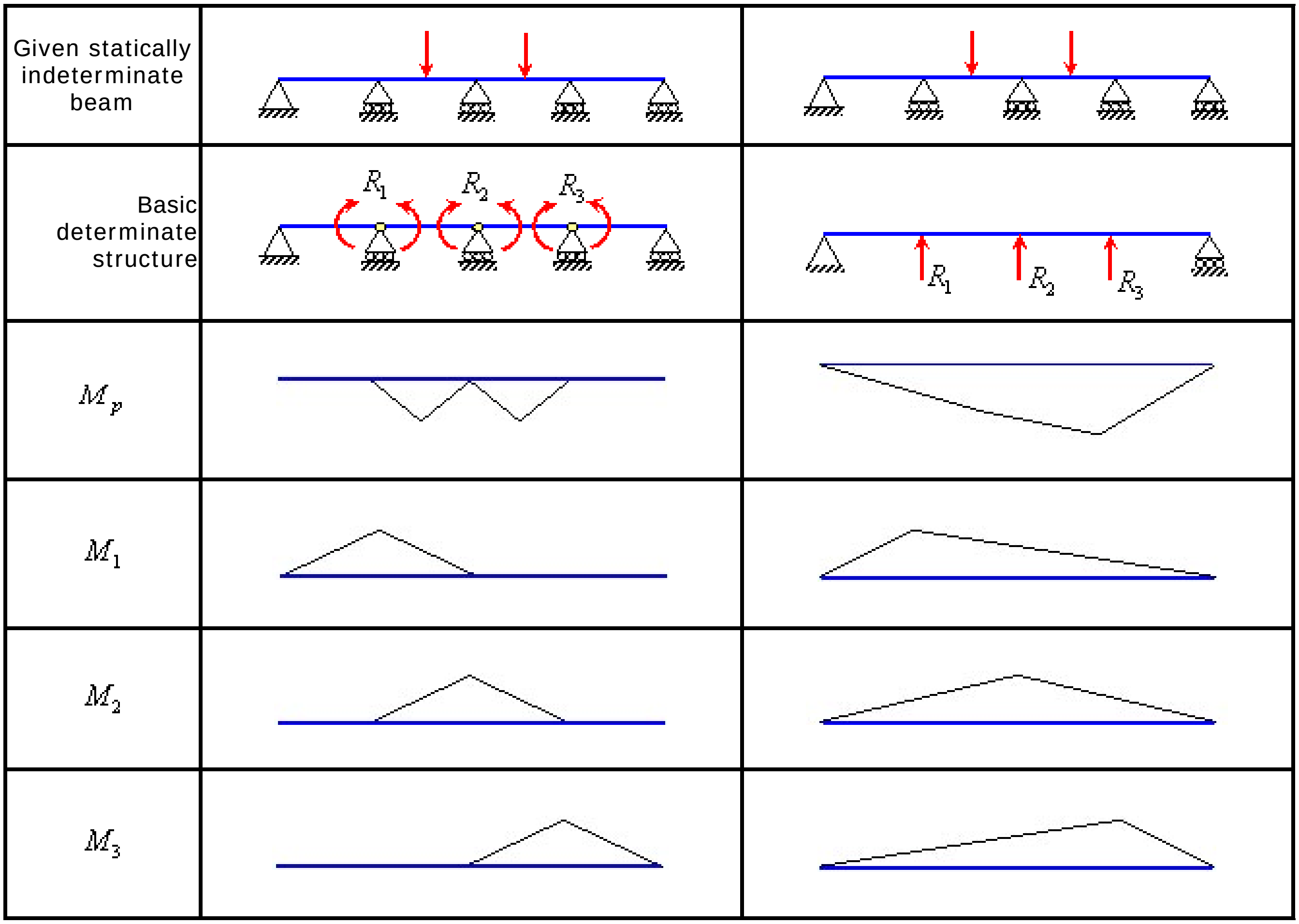


Figure 5.9

Example 5.7 Analyze the continuous beam shown in Figure 5.10(a) using the force method. Also, draw the bending moment diagram. *EI* is constant for entire beam.

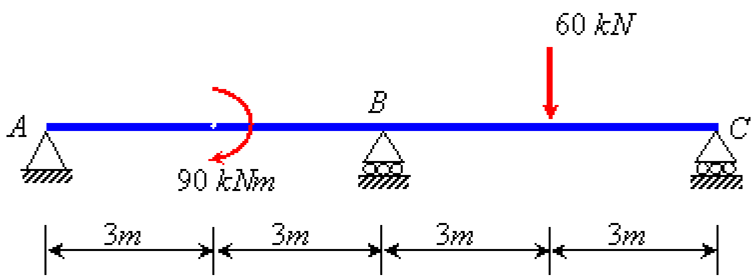


Figure 5.10(a)

Solution: The degree of static indeterminacy = 3–2 =1. The moment at *B* is taken as redundant *R* and the basic determinate structure will be then two simply supported beams as shown in Figure 5.10(b).

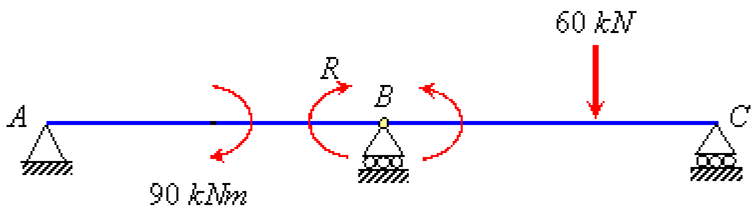


Figure 5.10(b)

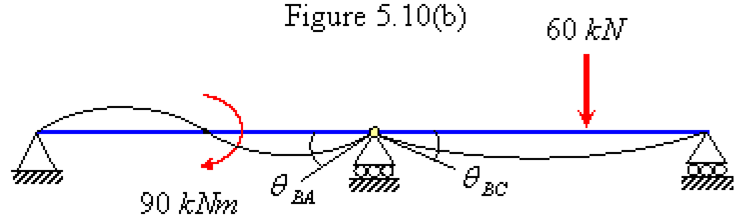


Figure 5.10(c)

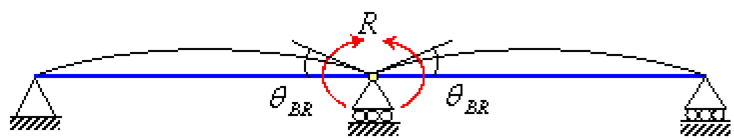


Figure 5.10(d)

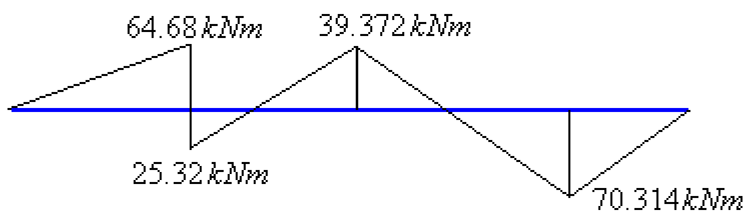


Figure 5.10(e)

Rotation of point *B* due to applied loads

$$= \theta_{BA} + \theta_{BC} \quad = \frac{90 \times 6}{24\,EI} + \frac{60 \times 6^2}{16\,EI} \quad = \frac{315}{2\,EI}$$

Rotation of point *B* due to *R*

$$= \frac{R \times 6}{3EI} + \frac{R \times 6}{3EI} = \frac{4R}{EI}$$

Equating the rotation of point *B* due to applied loads and *R* i.e.

$$\frac{4R}{EI} = \frac{315}{2\,EI}$$

or

$$R = 39.375 \text{ kNm}$$

The reaction at *A* is given by

$$V_A = \frac{90 + 39.375}{6} = 21.5625 \text{ kN} \ (\downarrow)$$

The vertical reaction at *C* is given by

$$V_C = \frac{60 \times 3 - 39.375}{6} = 23.4375 \text{ kN} \ (\uparrow)$$

The vertical reaction at *B* is

$$V_B = 60 + 21.5625 - 23.4375 = 58.125 \text{ kN}$$

The bending moment diagram of the beam is shown in Figure 5.10(e).

Example 5.8 Analyze the uniform continuous beam shown in Figure 5.11(a) using the force method. Also, draw the bending moment diagram.

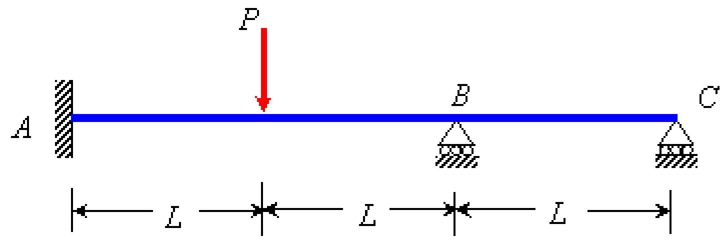


Figure 5.11(a)

Solution: The degree of static indeterminacy of the beam = 4 – 2 = 2. The moment at *A* and *B* are taken as unknown R_1 and R_2 , respectively.

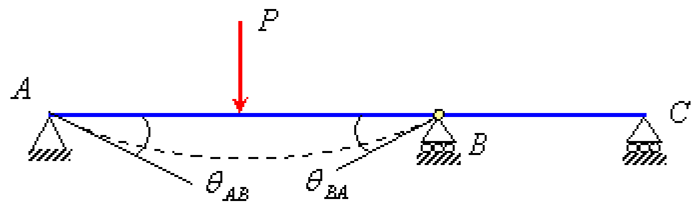


Figure 5.11(b) Basic determinate structure under the applied loads

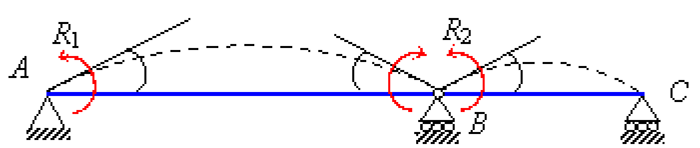


Figure 5.11(c) Basic structure under redundant R_1 and R_2

Equating the rotation at *A* due to applied loading and unknown R_1 and R_2 equal to zero i.e.

$$-\frac{P(2L)^2}{16EI}+\frac{R_1(2L)}{3EI}+\frac{R_2(2L)}{6EI}=0$$

or $$2R_1+R_2=\frac{3PL}{4} \qquad \text{(i)}$$

Equating the rotation at *B* due to applied loading and R_1 and R_2 equal to zero i.e.

$$-\frac{P(2L)^2}{16EI}+\frac{R_1(2L)}{6EI}+\frac{R_2(2L)}{3EI}+\frac{R_2(L)}{3EI}=0$$

or $$R_1+3R_2=\frac{3PL}{4} \qquad \text{(ii)}$$

Solving equations (i) and (ii)

$$R_1=\frac{3PL}{10} \qquad \text{and} \qquad R_2=\frac{3PL}{20}$$

The bending moment diagram of the beam is shown in Figure 5.11(d)

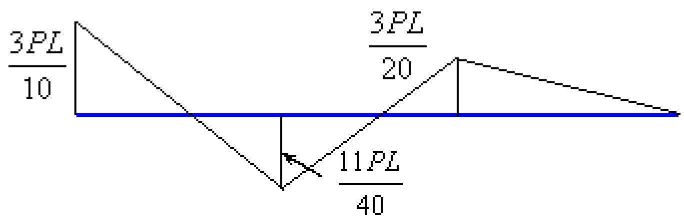


Figure 5.11(d)

Example 5.9 Find the force in various members of the pin-jointed frame shown in Figure 5.12(a). *AE* is constant for all members.

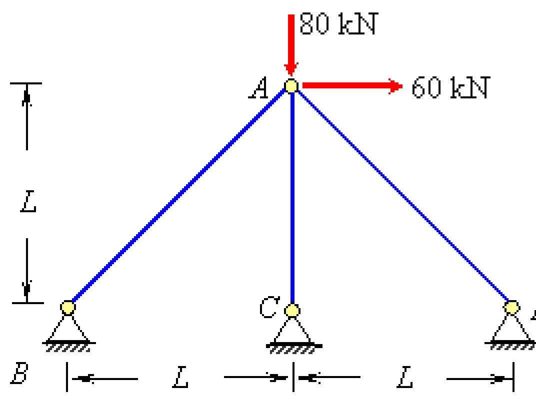


Figure 5.12(a)

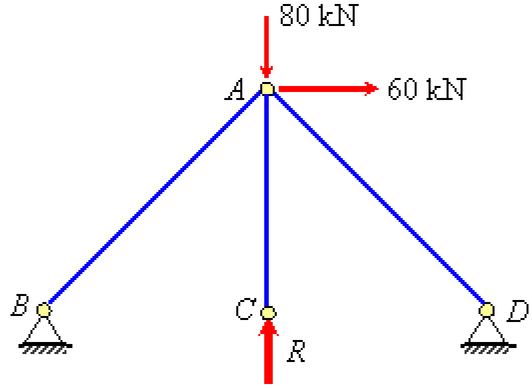


Figure 5.12(b) Basic determinate structure under the applied load and unknown R

Solution: The static indeterminacy of the pin-jointed frame = 1. The vertical reaction at *C* is taken as unknown force *R* . The computation of deflection of point *C* due to applied loading and *R* are shown in Tables 5.2 and 5.3, respectively.

Table 5.2

Member	Length	F_p	F_u	$F_u F_p L$
AB	$\sqrt{2}L$	$-20/\sqrt{2}$	$1/\sqrt{2}$	$-20L/\sqrt{2}$
AC	*L*	0	1	0
AD	$\sqrt{2}L$	$-140/\sqrt{2}$	$1/\sqrt{2}$	$-140L/\sqrt{2}$

		Σ	$-160L/\sqrt{2}$	

The vertical displacement of joint *C* due to applied loading = $-\dfrac{160L}{\sqrt{2}AE}$ (↓)

Table 5.3

Member	Length	F_p	F_u	F_uF_pL
AB	$\sqrt{2}L$	$R/\sqrt{2}$	$1/\sqrt{2}$	$RL/\sqrt{2}$
AC	*L*	*R*	1	*R*
AD	$\sqrt{2}L$	$R/\sqrt{2}$	$1/\sqrt{2}$	$RL/\sqrt{2}$
	Σ	$(\sqrt{2}+1)RL$		

The vertical displacement of joint *C* due to *R* = $\dfrac{(\sqrt{2}+1)RL}{AE}$ (↑)

Adding the displacement of point *C* due to applied loading and *R* and equating it to zero i.e.

$$\frac{(\sqrt{2}+1)RL}{AE}-\frac{160L}{\sqrt{2}AE}=0$$

$$\therefore \qquad R=\frac{80\sqrt{2}}{(\sqrt{2}+1)}\text{ kN}$$

The force in various members of the frame are as follows

$$F_{AB}=\frac{60-10\sqrt{2}}{(\sqrt{2}+1)}\quad \text{kN (Tensile)}$$

$$F_{AC}=\frac{80\sqrt{2}}{(\sqrt{2}+1)}\quad \text{kN (Compressive)}$$

$$F_{AD}=\frac{60+70\sqrt{2}}{(\sqrt{2}+1)}\quad \text{kN (Compressive)}$$

Example 5.10 Find the force in various members of the pin-jointed frame shown in Figure 5.13(a). *AE* is constant for all members.

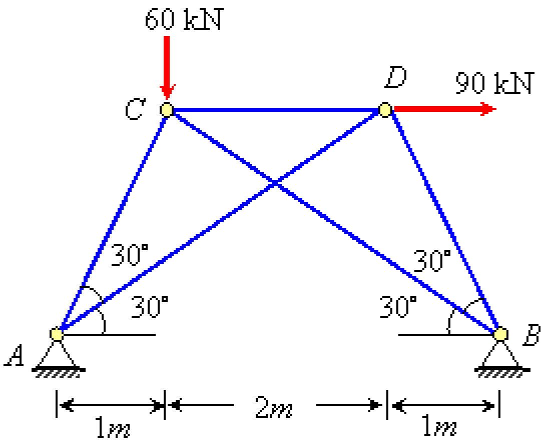


Figure 5.13(a)

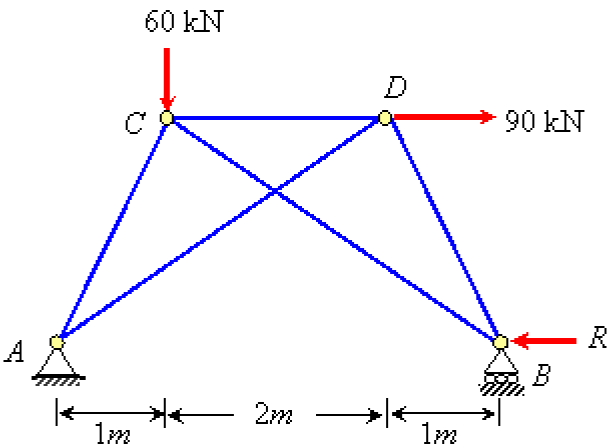


Figure 5.13(b) Basic determinate structure under the applied load and unknown R

Solution: The static indeterminacy of the pin-jointed frame = 1. The horizontal reaction at *B* is taken as unknown force *R* . The computation of horizontal deflection of point *B* due to applied loading and *R* are shown in Tables 5.4 and 5.5, respectively.

Table 5.4

Member	Length(m)	F_P	F_u	$F_u \; F_P L$
AC	2	-100.44	1	-200.88
AD	3.46	161.912	$-\sqrt{3}$	-970.32
CD	2	-96.967	2	-387.869
BC	3.46	53.97	$-\sqrt{3}$	-323.83
BD	2	-93.48	1	-186.96

Σ -2069.86

The horizontal displacement of joint *B* due to applied loading = $-\dfrac{2069.86}{AE}$ (→)

Table 5.5

Member	Length(m)	F_P	F_u	$F_u \; F_P L$
AC	2	*R*	1	2*R*
AD	3.46	$-\sqrt{3}\ R$	$-\sqrt{3}$	10.392 *R*
CD	2	2*R*	2	8 *R*
BC	3.46	$-\sqrt{3}\ R$	$-\sqrt{3}$	10.392 *R*

BD	2	*R*	1	2*R*
	Σ			32.784 *R*

The horizontal displacement of joint *B* due to *R* = $\frac{32.784\,R}{AE}$ ($\leftarrow$)

Adding the horizontal displacemet of point *B* due to applied loading and *R* and equating it to zero i.e.

$$\frac{32.784R}{AE} - \frac{2069.86}{AE} = 0$$

$\therefore$ R = 63.136 kN

The force in various members are as follows

F_{AC} = –37.305 kN

F_{AD} = 52.558 kN

F_{CD} = 29.328 kN

F_{BC} = –55.383 kN

F_{BD} = –30.345 kN

(–ve indicates compressive force and +ve indicates tensile force)

Example 5.11 Analyze the non-prismatic fixed beam shown in Figure 5.14(a) using force method.

Solution: Degree of indeterminacy of the system = 2. We choose shear force and moment at section *C* as redundant R_1 and R_2, respectively.

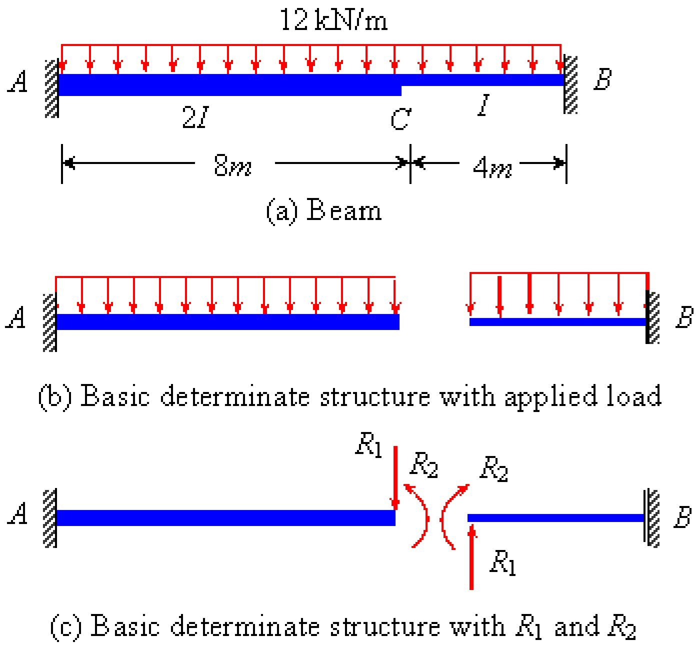


Figure 5.14 Beam of Example 5.11

Total displacement in the direction of R_1

$$\frac{R_1 \times 8^3}{3E(2I)} + \frac{R_1 \times 4^3}{3EI} - \frac{R_2 \times 8^2}{2E(2I)} + \frac{R_2 \times 4^2}{2EI} + \frac{12 \times 8^4}{8E(2I)} - \frac{12 \times 4^4}{8EI} = 0$$

or $$40R_1 - 3R_2 = -1008 \qquad \text{(i)}$$

Total rotation in the direction of R_2

$$-\frac{R_1 \times 8^2}{2E(2I)} + \frac{R_1 \times 4^2}{2EI} + \frac{R_2 \times 8}{E(2I)} + \frac{R_2 \times 4}{EI} - \frac{12 \times 8^3}{6E(2I)} - \frac{12 \times 4^3}{6EI} = 0$$

or $$-R_1 + R_2 = 80 \qquad \text{(ii)}$$

$$R_1 = -20.756 \text{ kN} \quad \text{and} \quad R_2 = 59.243 \text{ kNm}$$

The reactions at support are given by

$$V_A = 75.244 \text{ kN} \ (\uparrow)$$
$$M_A = 158.709 \text{ kNm} \ (\circlearrowright)$$
$$V_B = 68.756 \text{ kN} \ (\uparrow)$$
$$M_B = 119.781 \text{ kNm} \ (\circlearrowleft)$$

The bending moment diagram of the beam is shown in Figure 5.14(d)

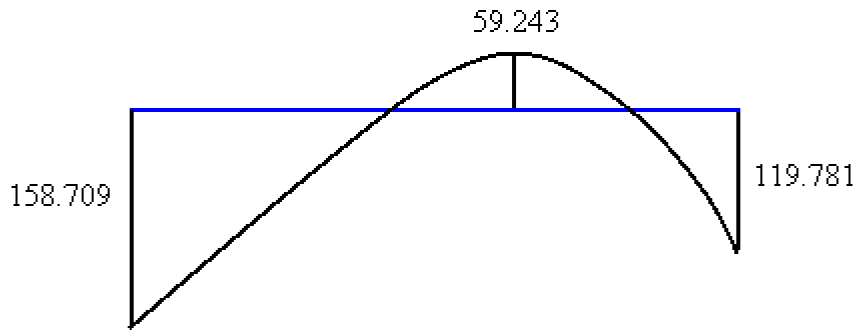


Figure 5.14(d) Bending moment diagram (kNm unit)

Example 5.12 Determine the horizontal thrust in a two-hinged trapezoidal arch. *EI* is constant.

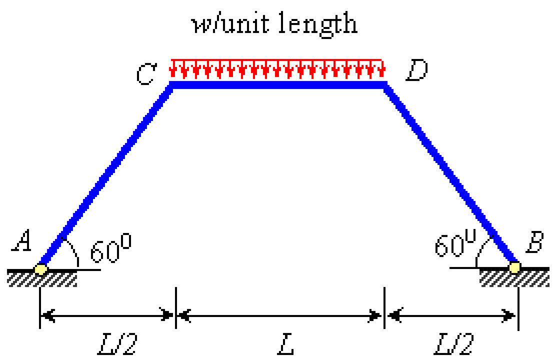


Figure 5.15(a) Given Frame

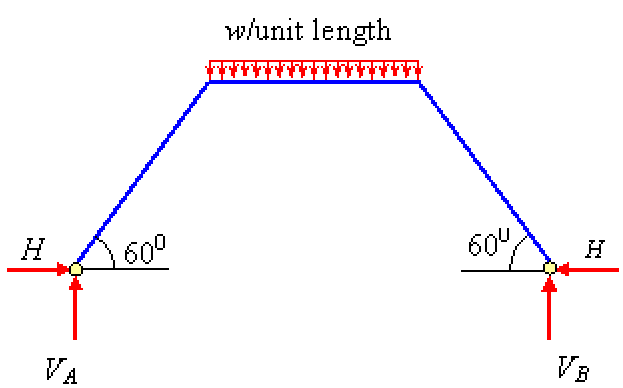


Figure 5.15(b) Basic determinate

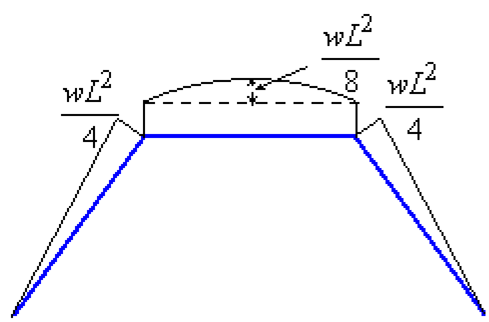


Figure 5.15(c) BMD due to applied loading

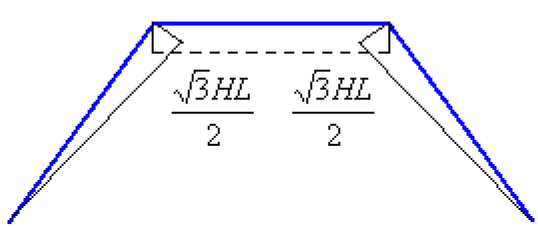


Figure 5.15(d) BMD due to H

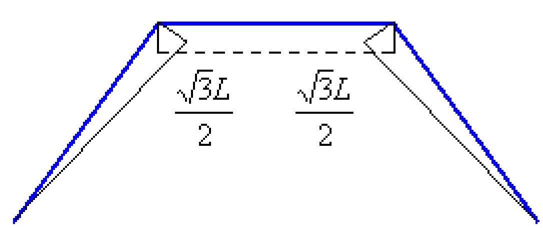


Figure 5.15(e) BMD due to $H = 1$

Solution: The degree of static indeterminacy = 4–3 = 1. Let *H* be the unknown horizontal reaction at *A* and *B* . Let V_A and V_B be the vertical reactions at support *A* and *B* respectively

Due to symmetry, $V_A = V_B = \frac{wL}{2}$

Displacement in the direction of *H* due to applied loading is calculated by multiplication of two bending moment diagrams of Figure 5.15(c) and (e).

$$= -\frac{1}{3} \times \frac{wL^2}{4} \times \frac{\sqrt{3}L}{2} \times \frac{L}{EI} - \frac{wL^2}{4} \times \frac{\sqrt{3}L}{2} \times \frac{L}{EI} - \frac{2}{3} \times \frac{wL^2}{8} \times \frac{\sqrt{3}L}{2} \times \frac{L}{EI} - \frac{1}{3} \times \frac{wL^2}{4} \times \frac{\sqrt{3}L}{2} \times \frac{L}{EI}$$

$$= -\frac{\sqrt{3}wL^4}{4EI}$$

Displacement in the direction of *H* due to *H* obtained by multiplying diagram of Figure 5.10(d) and (e).

$$= -\frac{1}{3} \times \frac{\sqrt{3}HL}{2} \times \frac{\sqrt{3}L}{2} \times \frac{L}{EI} + \frac{\sqrt{3}HL}{2} \times \frac{\sqrt{3}L}{2} \times \frac{L}{EI} + \frac{1}{3} \times \frac{\sqrt{3}HL}{2} \times \frac{\sqrt{3}L}{2} \times \frac{L}{EI}$$

$$= \frac{5HL^3}{4EI}$$

Since the net deflection in the direction of *H* is zero, therefore

$$-\frac{\sqrt{3}wL^4}{4EI} + \frac{5HL^3}{4EI} = 0$$

or $$H = \frac{\sqrt{3}wL}{5}$$

Example 5.13 Determine the reaction of the propped cantilever beam if the beam is assumed to be subjected to a linear temperature gradient such that the top surface of the beam is at temperature T_t and lower at T_b . The beam is uniform having flexural rigidity as *EI* and depth *d* . The coefficient of thermal expansion for beam material is α .

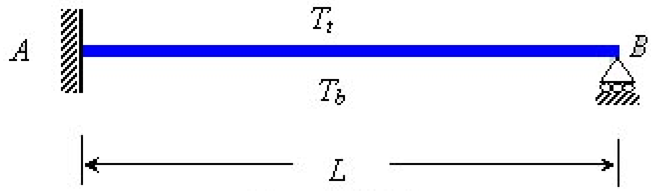


Figure 5.16(a)

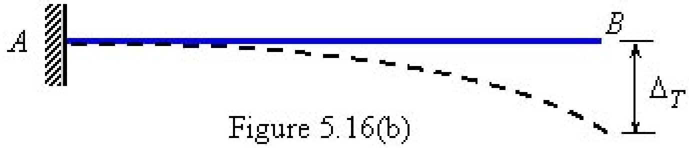


Figure 5.16(b)

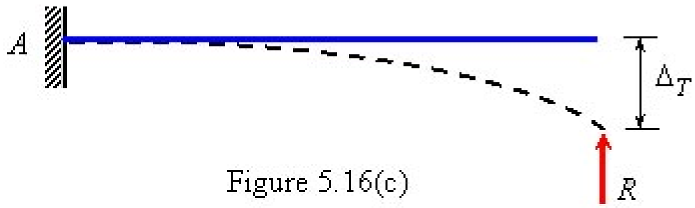


Figure 5.16(c)

Solution: The degree of static indeterminacy =1. Remove the support at *B* and allow the beam deflect freely under the temperature variation. The deflection of the free end of the beam due to temperature variation (from eqs 4.28 of Chapter 4).

$$\Delta_T = \frac{\alpha(T_t - T_b)L^2}{2d} \qquad \text{(i)}$$

Apply the force *R* at point *B* such that the deflection in the direction of *R* is equal to Δ_T . Since deflection of a cantilever beam due to force *R* is equal $RL^3/3EI$, therefore

$$\Delta_T = \frac{RL^3}{3EI} \qquad \text{(ii)}$$

Equating the Δ_T from two expressions of Eqs. (i) and (ii)

$$\frac{RL^3}{3EI} = \frac{\alpha(T_t - T_b)L^2}{2d}$$

$$\therefore \quad R = \frac{3EI\alpha(T_t - T_b)}{2dL}$$

The vertical reaction and bending moment at *A* will be

$$V_A = R = \frac{3EI\alpha(T_t - T_b)}{2dL} \quad (\downarrow)$$

$$M_A = RL = \frac{3EI\alpha(T_t - T_b)}{2d} \quad (\circlearrowleft)$$

Recap

In this course you have learnt the following

- Concept of force method for analysis of statically indeterminate structure.
- Selection of the basic determinate structure.
- Ellustration of force method by numerical examples.