Module 1 : Introduction : Review of Basic Concepts in Mechnics
Lecture 3 : Constitutive Relations and Compatibility Conditions

Objectives

In this course you will learn the following

- Review of the concept of constitutive relations.
- Stress-strain diagrams.
- Definitionof prameters related to material properties : - Modulus of elasticity, poison's ratio, shear modulus etc.
- Hooke's law.
- Stress and strain tensors.
- Review of the concepts of compatibility conditions; Interpretation as boundary conditions.

1.3 Constitutive Relations

Equilibrium equations help us obtain the forces that are acting, both internally and externally, at various parts of a body. However, for deformable solid bodies, understanding their deformation behaviour under the given stress/loading condition (based on the equilibrium) is of primary importance. The deformation behaviour in such a system is studied through various parameters, such as strain, displacement, rotation, etc. These deformation parameters are obtained based on the stress-strain relations of the material which the deformable solid is made of. These are known as Constitutive Relations and are material-specific. The stress-strain diagram for ductile steel (Figure 1.9) based on a tension test is an example of constitutive relations. It gives us a relation between the *engineering* (tensile) stress (σ) and engineering (tensile) strain ($\in$) for ductile steel at different stress (or strain) values.

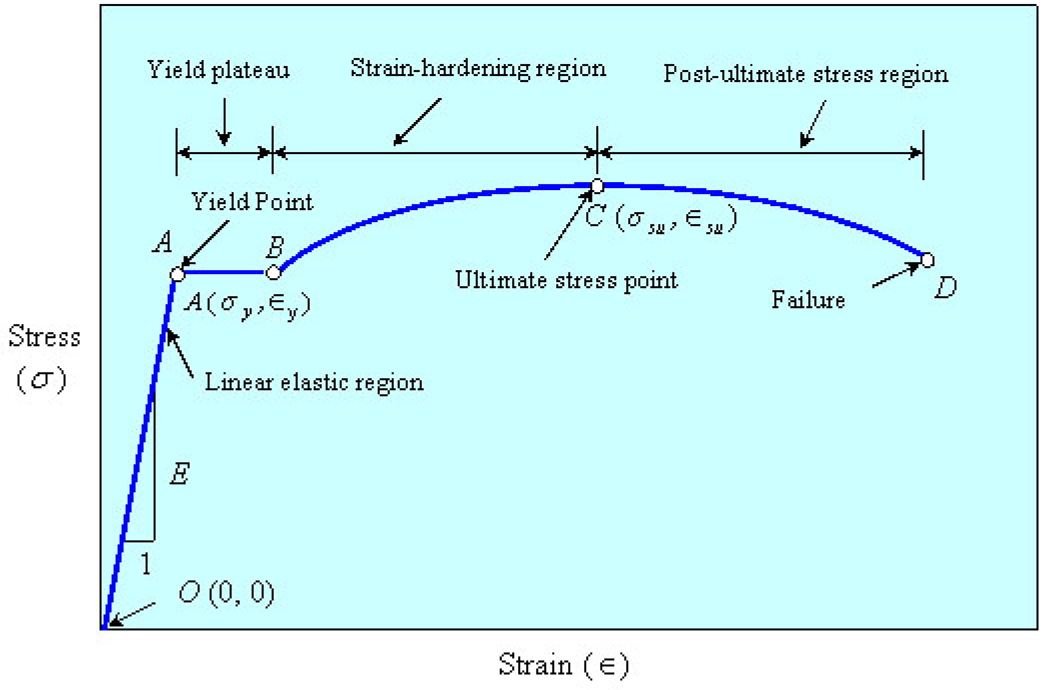

Figure 1.9 Stress-strain diagram for ductile steel

Similar stress-strain diagram can be obtained (through experiments) for different materials (aluminium, wood, tool steel, concrete, etc.) and for different types of deformation (uniaxial tensile and compressive, shear, transverse, dilatational, etc.). For the ease of use, these relations are idealized into simple mathematical rules. In Structural Mechanics, we will limit ourselves to *linear elastic isotropic homogeneous* materials only.

A material is called linear elastic if its stress-strain relation is linear and if when the material is unloaded it traces back the same stress-strain (loading) path. In other words, stress is a single-valued linear function of strain. The behaviour of ductile steel from point "*O*" to "*A*" (Figure 1.9) is a linear elastic one. A material will be isotropic if its constitutive relations are non-directional (same for any direction in space, *x* , *y* or *z*) and it will be homogeneous if it displays the same properties (e.g. a constitutive relation) at any point of the system (same properties at [x_1, y_1, z_1] and [x_2, y_2, z_2]). Some basic constitutive relations for a linear elastic isotropic homogeneous material are briefly discussed in the following sections.

1.3.1 Modulus of Elasticity

Hooke's Law provides us the relation for uniaxial stress

$$\sigma = E \epsilon \qquad \textbf{(1.5)}$$

The constant of proportionality is called the *elastic modulus* , *modulus of elasticity* or *Young's modulus* . Since ϵ is dimensionless the unit of *E* is same as that of uniaxial stress (e.g. N / mm^2).

1.3.2 Poison's Ratio

Uniaxial forces case strains not only in its direction, but also in the transverse/lateral directions. For a tensile strain in the axial direction, there will always be compressive strains in the lateral directions, and vice versa. Poisson's Ratio (ν) relates the lateral strains to the axial strain

$$\nu = -\frac{\text{lateral strain}}{\text{axial strain}} \qquad \textbf{(1.6)}$$

Note that this ratio is always a dimensionless positive number.

1.3.3 Coefficient of Linear Thermal Expansion

Linear thermal strain (ϵ_T) due to change in temperature (ΔT) is obtained by using this coefficient (α)

$$\epsilon_T = \alpha \Delta T \tag{1.7}$$

α has units of per degrees Centigrade (or Fahrenheit)

1.3.4 Shear Modulus

For shear stress (τ) and shear strain (γ), we have a constitutive relation similar to the Hooke's Law for linear stress and strain.

$$\tau = G\gamma \tag{1.8}$$

The constant of proportionality (*G*) is known as the *shear modulus* or *modulus of rigidity* . It has same units as modulus of elasticity (*E*). It can be proved that:

$$G = \frac{E}{2(1+\nu)} \tag{1.9}$$

1.3.5 Dilatation and Bulk Modulus

Dilatation (*e*) is defined as the change of volume per unit volume

$$e = \epsilon_x + \epsilon_y + \epsilon_z \tag{1.10}$$

If a three-dimensional body is subjected to uniform hydrostatic pressure *p* , then the ratio of this (compressive) pressure to the dilatation is known as the *bulk modulus* (*k*)

$$k = -\frac{p}{e} = \frac{E}{3(1-2\nu)} \tag{1.11}$$

k is also called the *modulus of compression* .

1.3.6 Generalized Hooke's Law

This is an extension of the Hooke's Law to three dimensions considering both linear and shears deformations. It is based on the generalized definitions of strain. As for the Hooke's Law for linear strain/deformation, the equations for Generalized Hooke's Law are applicable for linear elastic isotropic homogeneous materials only. The 6 equations for linear and shear strains are:

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} \tag{1.12a}$$

$$\varepsilon_y = -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} \tag{1.12b}$$

$$\varepsilon_z = -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{\sigma_z}{E} \tag{1.12c}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \tag{1.12d}$$

(1.12e)

$$\gamma_{yz} = \frac{\tau_{yz}}{G}$$

$$\gamma_{zx} = \frac{\tau_{zx}}{G} \qquad \textbf{(1.12f)}$$

which can also be expressed alternatively as expressions for stress:

$$\sigma_x = \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) + \mu\varepsilon_x \qquad \textbf{(1.13a)}$$

$$\sigma_y = \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) + \mu\varepsilon_y \qquad \textbf{(1.13b)}$$

$$\sigma_z = \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) + \mu\varepsilon_z \qquad \textbf{(1.13c)}$$

$$\tau_{xy} = G\gamma_{xy} \qquad \textbf{(1.13d)}$$

$$\tau_{yz} = G\gamma_{yz} \qquad \textbf{(1.13e)}$$

$$\tau_{zx} = G\gamma_{zx} \qquad \textbf{(1.13f)}$$

where λ and μ are the Lame parameters which are related to the Young's modulus *E* and Poisson's ratio ν :

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \mu = \frac{E}{(1+\nu)} \qquad \textbf{(1.14)}$$

These equations can also be expressed as relation between the stress and strain tensors

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} \qquad \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} \end{bmatrix}$$

Stress tensor **Strain tensor**

Note that, in a strain tensor, the shear strain (e.g. γ_{xy}) is replaced by the *pure* or irrotational shear strain ($\epsilon_{xy} = \frac{\gamma_{xy}}{2}$).

1.4 Compatibility Conditions

Compatibility conditions represent restriction on deformations at specific locations in a system. The location can be both inside the system and at its boundary. The deformations in a system have to be compatible with the geometry of the surrounding (both external and internal), and this compatibility is assured through these conditions. In other words, compatibility conditions specify that deformations in a member/part of a system have to be compatible with the support conditions (external), as well as with other members/parts of the system (internal). For example, in the case of bar *ABC* in (Figure 1.10), various compatibility conditions on horizontal displacements are:

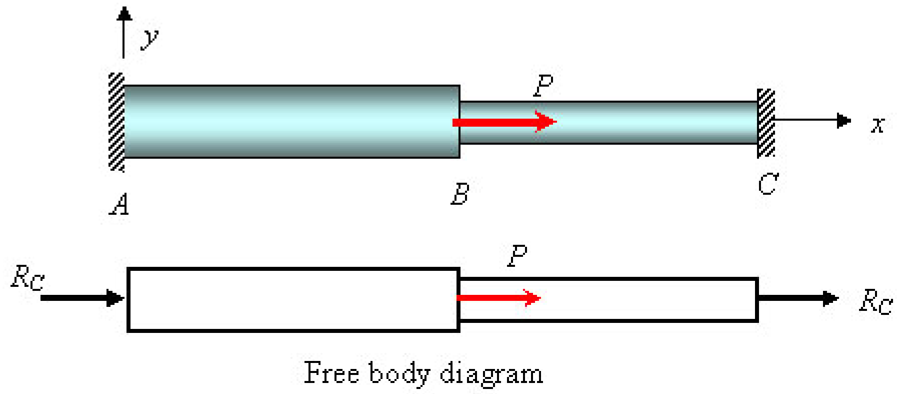

Figure. 1.10 Axially loaded bar *ABC*

$$\Delta_{AB_A} = 0 \tag{1.15}$$

$$\Delta_{BC_C} = 0 \tag{1.16}$$

$$\Delta_{AB_B} = \Delta_{BC_B} \tag{1.17}$$

Where Δ_{AB_B} is the deflection of bar *AB* at point *B*.

The deformation behaviour of a structural element is usually expressed through differential equations and the associated compatibility conditions are represented as boundary conditions for those equations.

Recap

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- Hooke's law.
- Stress and strain tensors.
- Review of the concepts of compativility conditions; Interpretation as boundary conditions.