Module 5 : Force Method - Introduction and applications

Lecture 1 : Analysis of Statically Indeterminate Beams

Objectives

In this course you will learn the following

- Introduction to statically indeterminate structure.
- Analysis of statically indeterminates beam using moment area and conjugate beam method.
- To demonstrate the application of moment area and conjugate beam method through illustrative examples.

5.1 Introduction

A strucure in which the laws of statics are not sufficient to determine all the unknown forces or moments is said to be statically indeterminate. Such structures are analyzed by writing the appropriate equations of static equilibrium and additional equations pertaining to the deformation and constraints known as compatibility condition.

The statically indeterminate structures are frequently used for several advantages. They are relatively more economical in the requirement of material as the maximum bending moments in the structure are reduced. The statically indeterminate are more rigid leading to smaller deflections. The disadvantage of the indeterminate structure is that they are subjected to stresses when subjected to temperature changes and settlements of the support. The construction of indeterminate structure is more difficult if there are dimensional errors in the length of members or location of the supports.

This chapter deals with analysis of statically indeterminate structures using various force methods.

5.2 Analysis of Statically Indeterminate Beams

The moment area method and the conjugate beam method can be easily applied for the analysis of statically indeterminate beams using the principle of superposition. Depending upon the degree of indeterminacy of the beam, designate the excessive reactions as redundant and modify the support. The redundant reactions are then treated as unknown forces. The redundant reactions should be such that they produce the compatible deformation at the original support along with the applied loads. For example consider a propped cantilever beam as shown in Figure 5.1(a). Let the reaction at *B* be *R* as shown in Figure 5.1(b) which can be obtained with the compability condition that the downward vertical deflection of *B* due to applied loading (i.e. Δ_0 shown in Figure 5.1(c)) should be equal to the upward vertical deflection of *B* due to *R* (i.e. Δ_0 shown in Figure 5.1(d)).

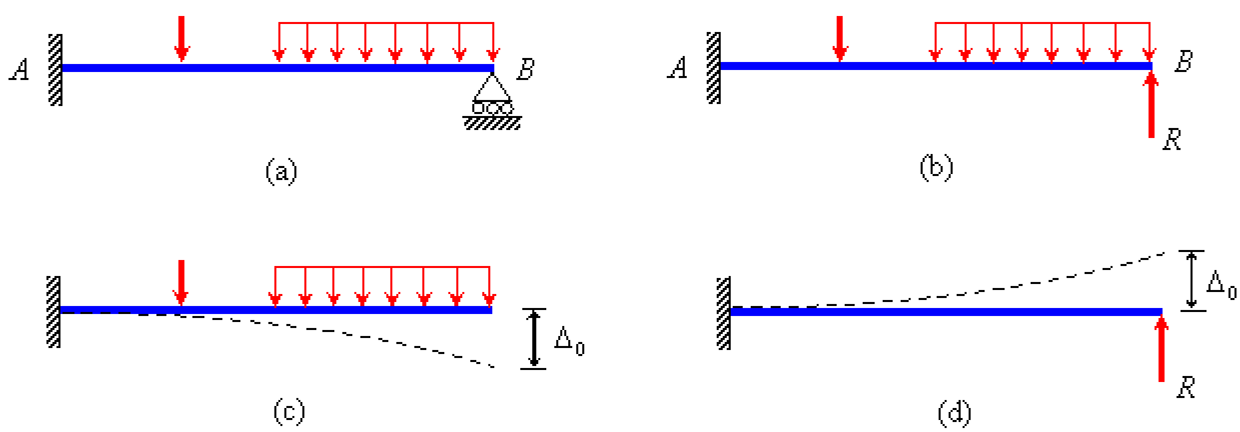


Figure 5.1

Example 5.1 Determine the support reactions of the propped cantilever beam as shown in Figure 5.2(a).

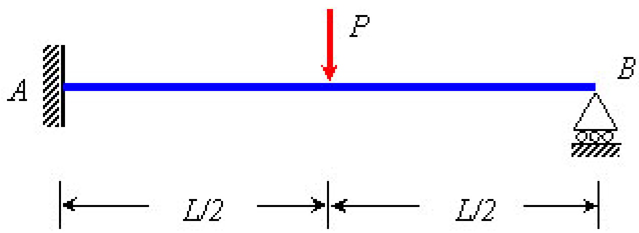


Figure 5.2(a)

Solution: The static indeterminacy of the beam is = 3 – 2 = 1. Let reaction at *B* is *R* acting in the upward direction as shown in Figure 5.2(b). The condition available is that the $\Delta_B = 0$.

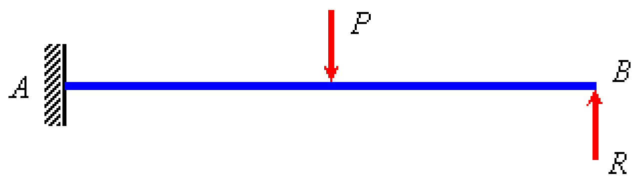
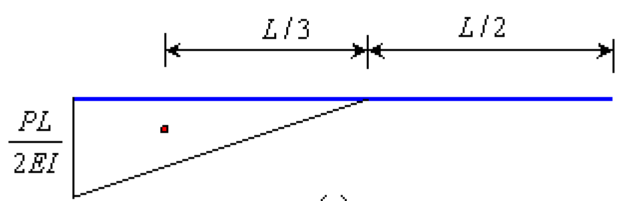


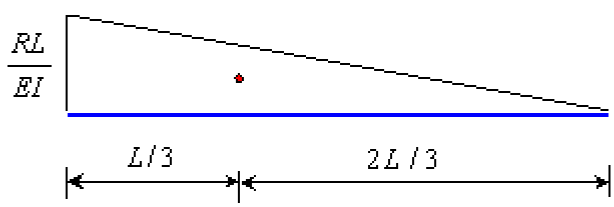
Figure 5.2(b)

(a) ***Moment area method***

The bending moment diagrams divided by *EI* of the beam are shown due to *P* and *R* in Figures 5.2(c) and (d), respectively.



(c)



(d)

Figure 5.2(c-d)

Since in the actual beam the deflection of the point *B* is zero which implies that the deviation of point *B* from the tangent at *A* is zero. Thus,

$$t_{BA} = 0$$

or

$$-\frac{1}{2}\times\frac{L}{2}\times\frac{PL}{2EI}\left(\frac{L}{2}+\frac{L}{3}\right)+\frac{1}{2}\times L\times\frac{RL}{EI}\left(\frac{2L}{3}\right)=0$$

$$\therefore \quad R = \frac{5P}{16} \quad A_{m_1} = \frac{AE}{L}\left[(D_{m_3} - D_{m_1})C_x + (D_{m_4} - D_{m_2})C_y\right]$$

Taking moment about *A* , the moment at *A* is given by

$$M_A = P \times \frac{L}{2} - \frac{5P}{16} \times L = \frac{3PL}{16}$$

The vertical rection at *A* is

$$V_A = P - \frac{5P}{16} = \frac{11P}{16}$$

The bending moment diagram of the beam is shown in Figure 5.2(e).

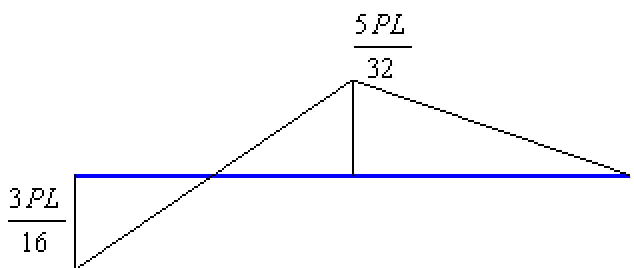


Figure 5.2(e)

(b) ***Conjugate beam method***

The corresponding conjugate beam of the propped cantilever beam and loading acting on it are shown in Figure 5.2(f).

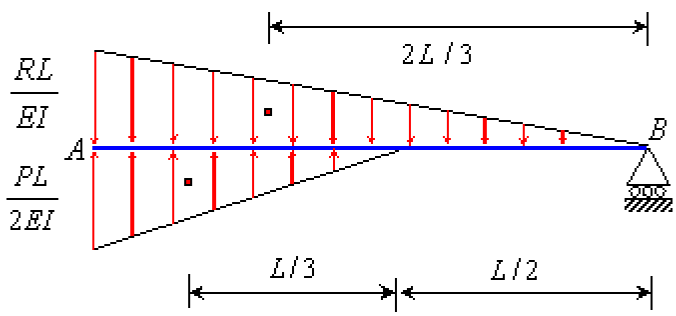


Figure 5.2(f)

The unknown *R* can be obtained by taking moment about *B* i.e.

$$-\frac{1}{2} \times \frac{L}{2} \times \frac{PL}{2EI}\left(\frac{L}{2} + \frac{L}{3}\right) + \frac{1}{2} \times L \times \frac{RL}{EI}\left(\frac{2L}{3}\right) = 0$$

$$\therefore \quad R = \frac{5P}{16}$$

Example 5.2 Determine the support reactions of the fixed beam with internal hinge as shown in Figure 5.3(a).

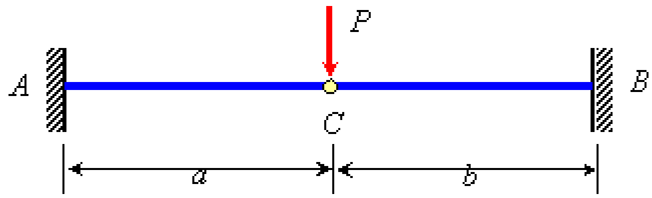


Figure 5.3(a)

Solution: The static indeterminacy of the beam is = 4-2-1 =1 Let the shear in the internal hinge be *R* . The free body diagrams of the two separated portions of the beam are shown in Figure 5.3(b) along with their *M/EI* diagrams. The unknown *R* can be obtained with the condition that the vertical deflection of the free ends of the two separated cantilever beams is identical.

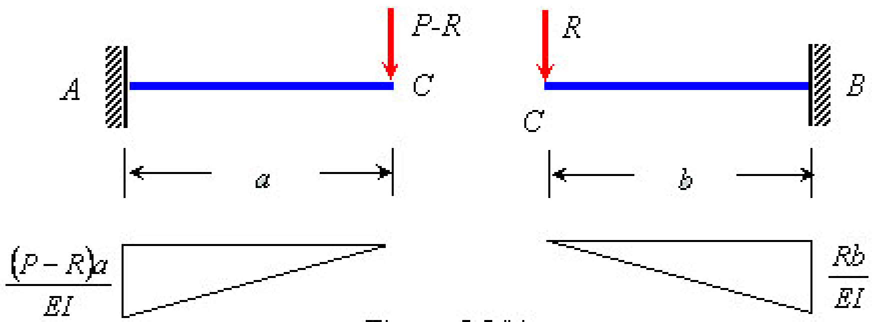


Figure 5.3(b)

Consider AC : The vertical displacement of *C* is given by

$$\Delta_C = t_{CA} = -\frac{1}{2} \times a \times \frac{(P-R)a}{EI} \times \frac{2a}{3}$$

or

$$\Delta_C = -\frac{(P-R)a^3}{3EI}$$

Consider CB : The vertical displacement of *C* is given by

$$\Delta_C = t_{CB} = -\frac{1}{2} \times b \times \frac{Rb}{EI} \times \frac{2b}{3}$$

$$\Delta_C = -\frac{Rb^3}{3EI}$$

Equating the Δ_C from Eqs. (i) and (ii)

$$-\frac{(P-R)a^3}{3EI} = -\frac{Rb^3}{3EI}$$

Solving for *R* will give

$$R = \frac{Pa^3}{(a^3+b^3)}$$

The reactions at the supports are given by

$$V_A = \frac{Pb^3}{(a^3 + b^3)} \quad (\uparrow) \qquad M_A = \frac{Pb^3 a}{(a^3 + b^3)} \quad (\circlearrowright)$$

$$V_B = \frac{Pa^3}{(a^3 + b^3)} \quad (\uparrow) \qquad M_B = \frac{Pa^3 b}{(a^3 + b^3)} \quad (\circlearrowleft)$$

Example 5.3 Determine the support reactions of the fixed beam with one end fixed and other supported on spring as shown in Figure 5.4(a). The stiffness of spring is $k = EI/L^3$.

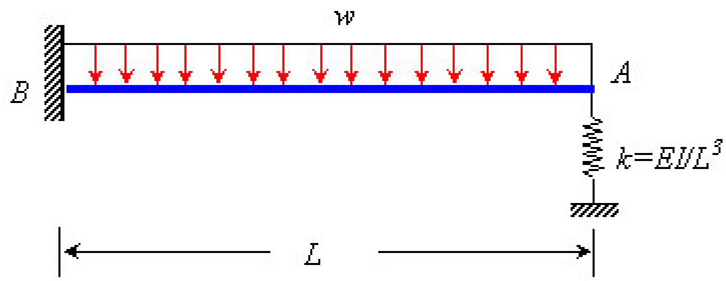


Figure 5.4(a)

Solution: The static indeterminacy of the beam is = 3–2 =1. Let the force in the spring be *R* . The free body diagram of the beam along with the *M/EI* diagram and spring are shown in Figure 5.4(b) and (c), respectively. The unknown *R* can be obtained with the condition that the vertical deflection of the free end of the beam and spring is identical.

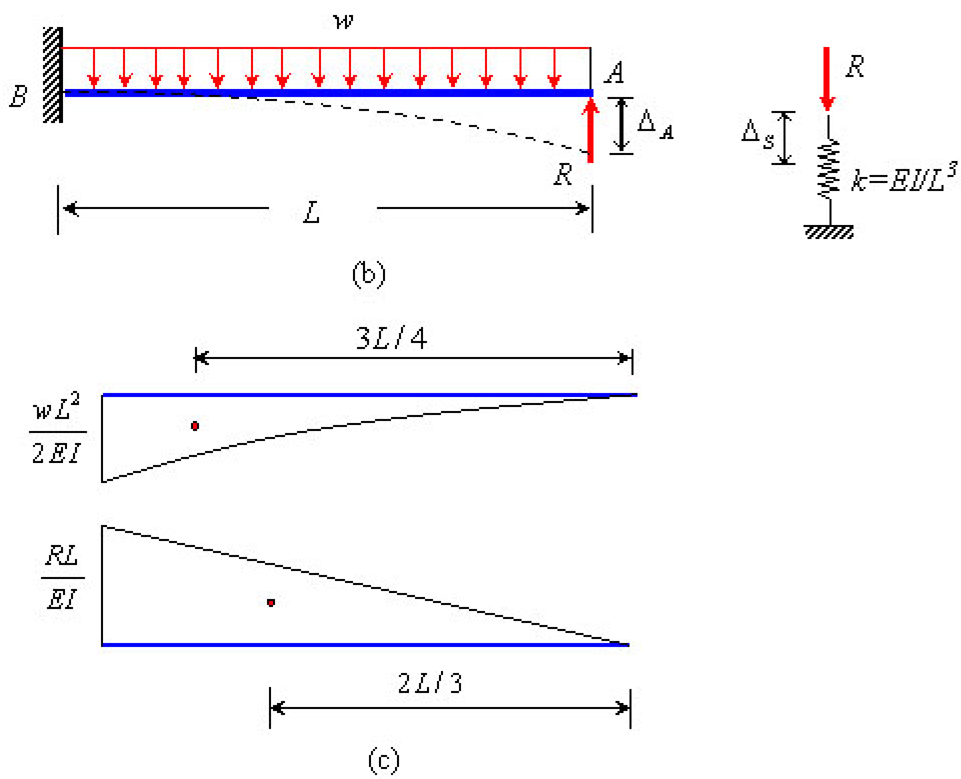


Figure 5.4(b)-(c)

Using moment area theorem, the deflection of free end *A* of the beam is

$$\Delta_A = -t_{A/B} = \left(\frac{1}{3}\times\frac{wL^2}{2EI}\times L\right)\times\frac{3L}{4} - \frac{1}{2}\times\frac{RL}{EI}\times L\times\frac{2L}{3}$$

$$\Delta_A = \frac{wL^4}{8EI} - \frac{RL^3}{3EI}$$

The downward deflection of spring is

$$\Delta_S = \frac{R}{k} = \frac{RL^3}{EI}$$

Equating Δ_A and Δ_S

$$\frac{RL^3}{EI} = \frac{wL^4}{8EI} - \frac{RL^3}{3EI}$$

$$\therefore \quad R = \frac{3wL}{32}$$

The bending moment at *B*

$$M_B = w\times L\times\frac{L}{2} - \frac{3wL}{32}\times L$$

$$= \frac{13wL^2}{32} \quad (\circlearrowleft)$$

The vertical reaction at *B*

$$V_B = w\times L - \frac{3wL}{32}$$

$$= \frac{29wL}{32} \quad (\uparrow)$$

The force in the spring $= \frac{3wL}{32}$ (compressive)

The bending moment diagram of the beam is shown in Figure 5.4(d).

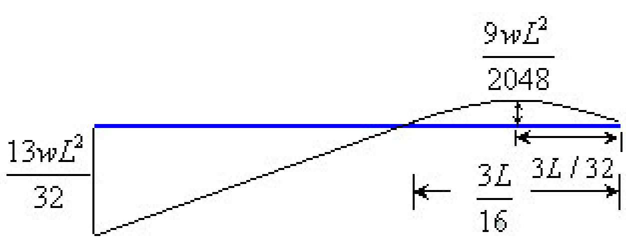


Figure 5.4(d)

Example 5.4 Determine the support reactions of the fixed beam as shown in Figure 5.5(a). The beam carries a uniformly distributed load, *w* over the left half span.

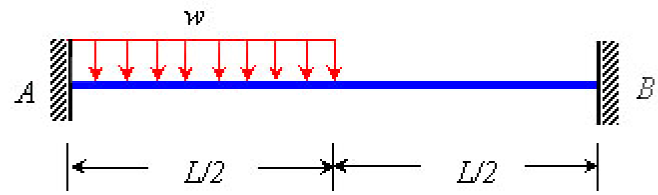


Figure 5.5(a)

Solution: The static indeterminacy of the beam is = 4-2 =2. Let the reactions at *B* be the unknown as shown in Figure 5.5(b).

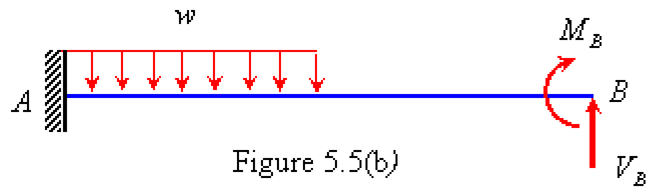


Figure 5.5(b)

(a) Moment Area Method

The free body diagram of the beam is shown below along with their *M/EI* diagrams. The unknowns V_B and M_B can be obtained with the condition that the vertical deflection and slope at *B* are zero.

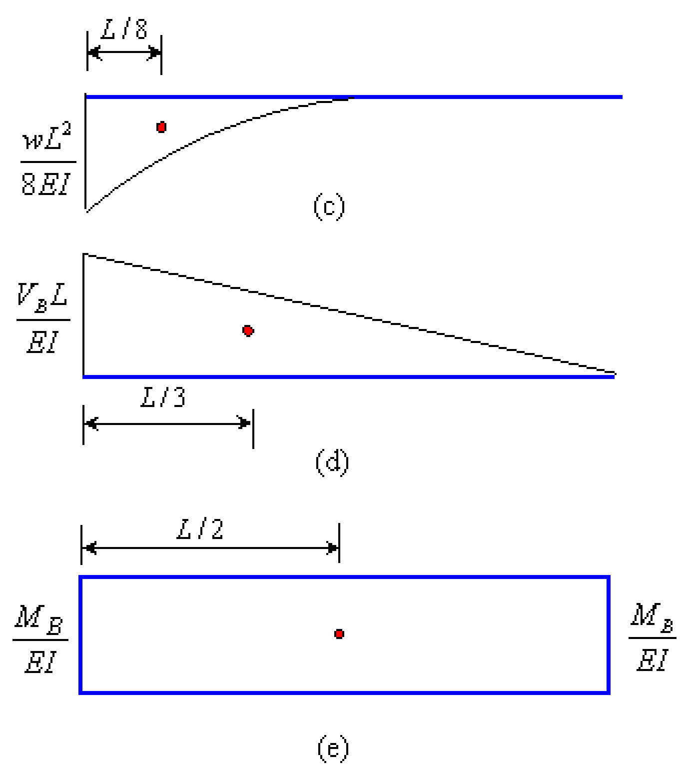


Figure 5.5 *M/EI* diagram due to (c) applied external load, (d) V_B and (e) due to M_B

Since the change of slope between points *A* and *B* is zero (due to fixed supports at *A* and *B*), therefore, according to the first moment area theorem,

$$\Delta\theta_{BA} = 0$$

$$\Delta\theta_{BA} = \left(-\frac{1}{3}\times\frac{L}{2}\times\frac{wL^2}{8EI}\right) + \left(\frac{1}{2}\times L\times\frac{V_B L}{EI}\right) - \left(\frac{M_B}{EI}\times L\right) = 0$$

or $$\frac{V_B L}{2} - M_B = \frac{wL^2}{48} \qquad \text{(i)}$$

$$t_{AB} = 0$$

$$t_{AB} = \left(-\frac{1}{3}\times\frac{L}{2}\times\frac{wL^2}{8EI}\right)\times\frac{L}{8} + \left(\frac{1}{2}\times L\times\frac{V_B L}{EI}\right)\times\frac{L}{3} - \left(\frac{M_B}{EI}\times L\right)\times\frac{L}{2} = 0$$

or $$\frac{V_B L}{6} - \frac{M_B}{2} = \frac{wL^2}{384} \qquad \text{(ii)}$$

Solving equations (i) and (ii)

$$V_B = \frac{3}{32}wL \;(\uparrow) \qquad and\; M_B = \frac{5}{192}wL^2 \;(\circlearrowright)$$

$$V_A = \frac{13}{32}wL \;(\uparrow) \qquad and\; M_A = \frac{11}{192}wL^2 \;(\circlearrowleft)$$

The bending moment diagram of the beam is shown in Figure 5.5(f)

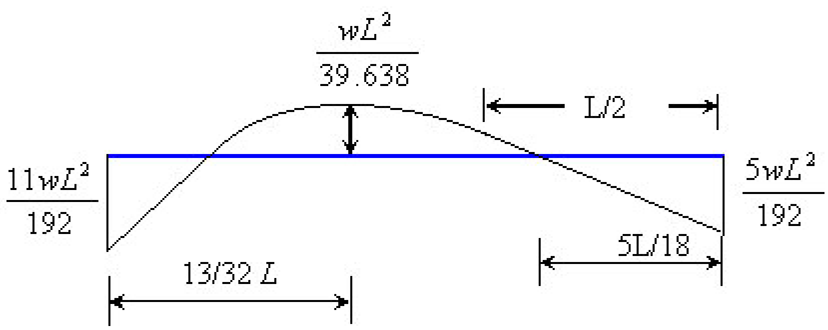


Figure 5.5(f)

(b) Conjugate Beam Method

The corresponding conjugate beam (i.e. free-free beam) and loading on it are shown in Figure 5.5(g).

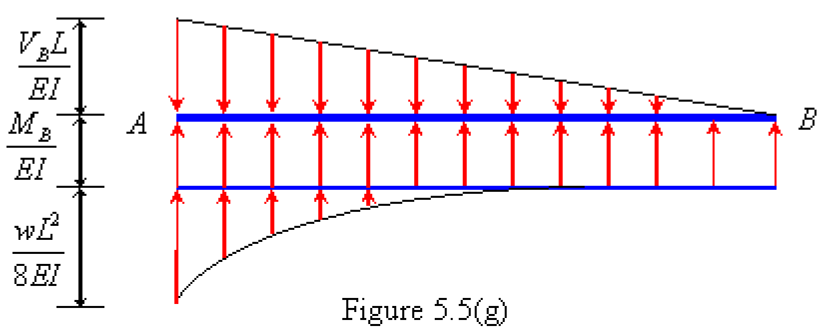


Figure 5.5(g)

Considering vertical equilibrium of all forces acting on Conjugate beam

$$\left(\frac{1}{2}\times L\times\frac{V_B L}{EI}\right) - \left(\frac{M_B}{EI}\times L\right) - \left(\frac{1}{3}\times\frac{L}{2}\times\frac{wL^2}{8EI}\right) = 0$$

or $$\frac{V_B L}{2} - M_B = \frac{wL^2}{48} \qquad \text{(iii)}$$

Taking moment about *A*

$$\left(\frac{1}{2}\times L\times\frac{V_B L}{EI}\right)\times\frac{L}{3}-\left(\frac{M_B}{EI}\times L\right)\times\frac{L}{2}-\left(\frac{1}{3}\times\frac{L}{2}\times\frac{wL^2}{8EI}\right)\times\frac{L}{8}=0$$

or $$\frac{V_B L}{6}-\frac{M_B}{2}=\frac{wL^2}{384} \qquad \text{(iv)}$$

Solving eqs. (iii) and (iv)

$$V_B=\frac{3wL}{32}$$

$$M_B=\frac{5wL^2}{192}$$

Example 5.5 The end *B* of a uniform fixed beam sinks by an amount D . Determine the end reactions using moment area method.

Solution: The degree of indeterminacy is 2. Let end reactions due to settlement at *B* be V_B and M_B as shown in Figure 5.6(b). The *M/EI* diagram of the beam is shown in Figure 5.6(c).

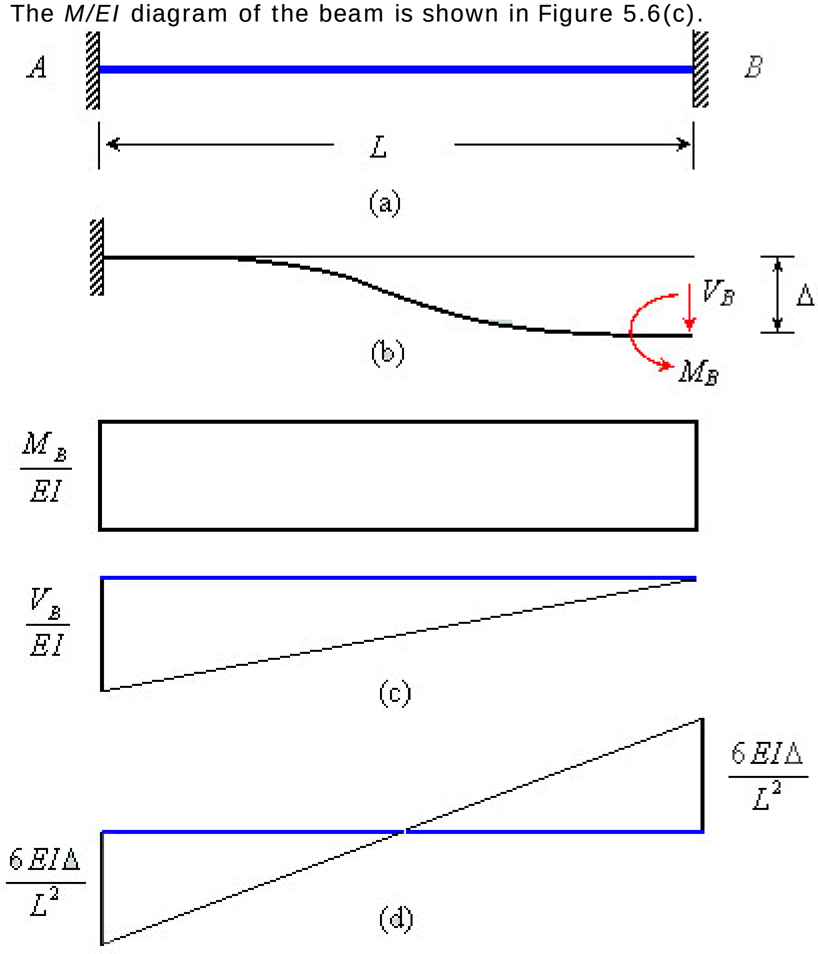


Figure 5.6(a)-(d)

Applying first moment area theorem between *A* and *B*

$$\Delta\theta_{AB}=L\times\frac{M_B}{EI}-\frac{1}{2}\times\frac{V_B L}{EI}\times L=0$$

or $$M_B=\frac{V_B L}{2} \qquad \text{(i)}$$

Applying second moment area theorem between point *A* and *B*

$$t_{BA} = \frac{M_B}{EI} \times L \times \frac{L}{2} - \frac{1}{2} \times \frac{V_B L}{EI} \times L \times \frac{2L}{3}$$

or

$$-\Delta = \frac{M_B L^2}{2EI} - \frac{V_B L^3}{3EI}$$

Solving eqs. (i) and (ii)

$$M_B = \frac{6EI\Delta}{L^2} \quad \text{and} \quad V_B = \frac{12EI\Delta}{L^3}$$

By equilibrium conditions, the reactions at support *A* are

$$M_A = \frac{6EI\Delta}{L^2} \; (\circlearrowleft) \quad \text{and} \quad V_A = \frac{12EI\Delta}{L^3} \; (\uparrow)$$

Example 5.6 Determine the support reactions of the continuous beam as shown in Figure 5.7(a).

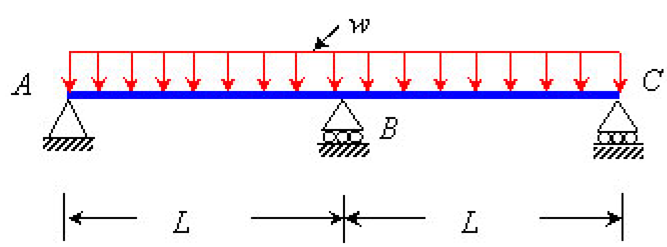


Figure 5.7(a)

Solution: The static indeterminacy of the beam is = 3-2 =1. Let the vertical reaction at *B* be the unknown *R* as shown in Figure 5.7(b). The *M/EI* diagrams of the beam are shown in Figure 5.7(c).

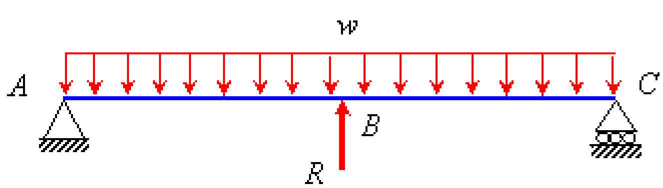


Figure 5.7(b)

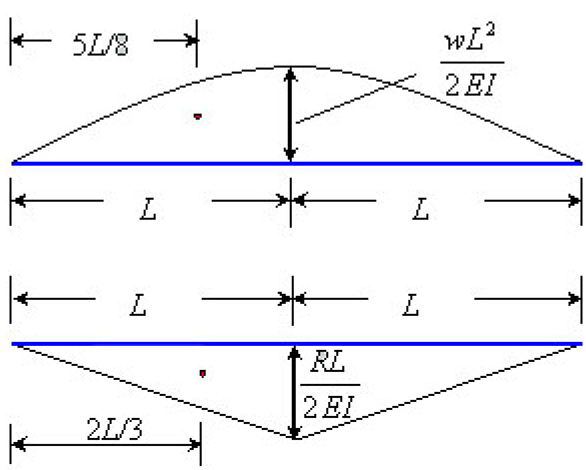


Figure 5.7(c)

Because of symmetry of two spans the slope at *B* , $\theta_B = 0$. As a result

$$t_{AB} = 0$$

or
$$t_{AB} = \left(\frac{2}{3} \times \frac{wL^2}{2EI} \times L\right)\frac{5L}{8} - \left(\frac{1}{2} \times \frac{RL}{2EI} \times L\right)\frac{2L}{3} = 0$$

or
$$R = \frac{5wL}{4}$$

The vertical reaction at *A* and *C* are

$$V_A = V_C = wL - \frac{1}{2} \times \frac{5wL}{4}$$
$$= \frac{3}{8} wL \ (\uparrow)$$

The bending moment diagram of the beam is shown in Figure 5.7(d).

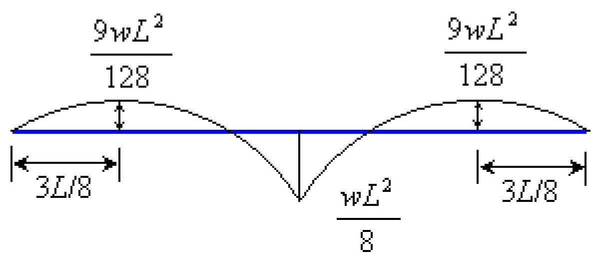


Figure 5.7(d)

Recap

In this course you have learnt the following

- Introduction to statically indeterminate structure.
- Analysis of statically indeterminates beam using moment area and conjugate beam method.
- To demonstrate the application of moment area and conjugate beam method through illustrative examples.