Module 4 : Deflection of Structures

Lecture 4 : Strain Energy Method

Objectives

In this course you will learn the following

- Deflection by strain energy method.
- Evaluation of strain energy in member under different loading.
- Application of strain energy method for different types of structure.

4.5 Deflection by Strain Energy Method

The concepts of strain, strain-displacement relationships are very useful in computing energy-related quantities such as work and strain energy. These can then be used in the computation of deflections. In the special case, when the structure is linear elastic and the deformations are caused by external forces only, (the complementary energy *U* * is equal to the strain energy *U*) the displacement of structure in the direction of force P_j is expressed by

$$\Delta_j = \frac{\partial U}{\partial P_j} \qquad \textbf{(4.16)}$$

This equation is known as Castigliano's theorem. It must be remembered that its use is limited to the calculation of displacement in linear elastic structures caused by applied loads. The use of this theorem is equivalent to the virtual work transformation by the unit-load theorem.

4.5.1 Calculation of Strain Energy

When external loads are applied on an elastic body they deform. The work done is transformed into elastic strain energy *U* that is stored in the body. We will develop expressions for the strain energy for different types of loads.

Axial Force **:** Consider a member of length *L* and axial rigidity *AE* subjected to an axial force *P* applied gradually as shown in the Figure 4.24. The strain energy stored in the member will be equal to the external work done by the axial force i.e

$$U = \frac{1}{2} \times P \times \Delta = \frac{1}{2} \times P \times \frac{PL}{AE} = \frac{P^2 L}{2AE} \qquad \textbf{(4.17)}$$

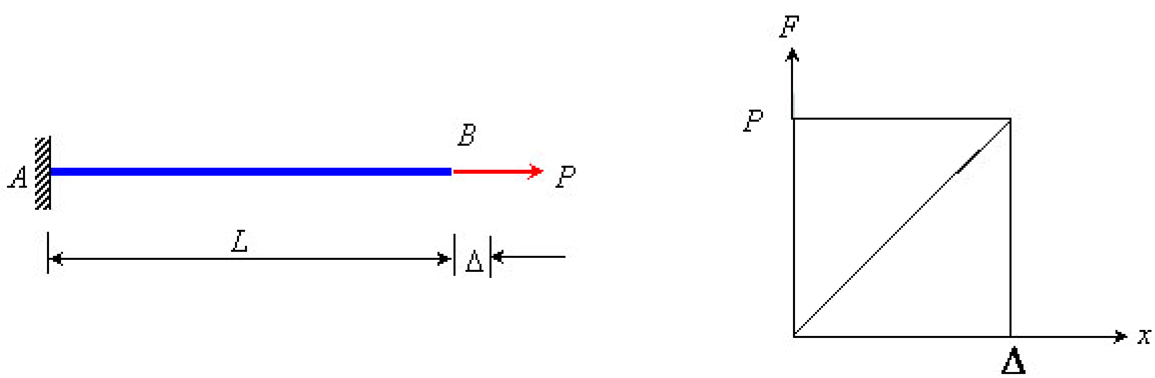

Figure 4.24 Member subjected to axial force

Bending Moment: Consider a beam of length *L* and flexural rigidity *EI* subjected to a general loading as shown in Figure 4.25. Consider a small differential element of length, *dx* . The energy stored in the small element is given by

$$dU = \frac{1}{2} \times M \times d\theta = \frac{1}{2} \times M \times \frac{M}{EI} dx = \frac{M^2}{2EI} dx \quad \textbf{(4.18)}$$

The total strain energy in the entire beam will be

$$U = \int_0^L \frac{M^2}{2EI} dx \quad \textbf{(4.19)}$$

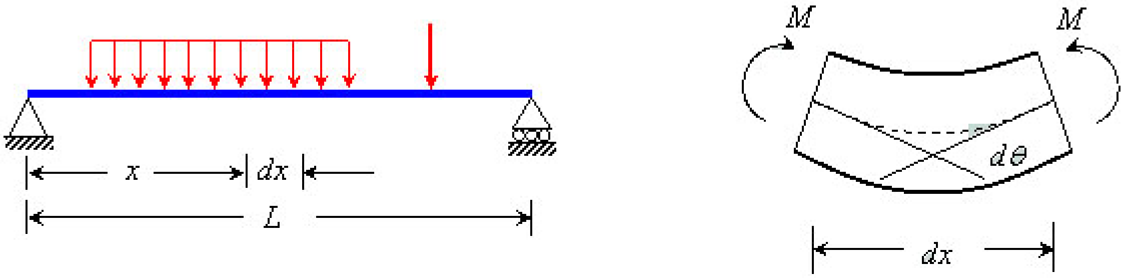

Figure 4.25 Member under bending

Shear Force: The strain energy stored in the member due to shearing force is expressed by

$$U = \int_0^L \frac{V^2}{2GA_s} dx \quad \textbf{(4.20)}$$

where *V* is the shearing force; and GA_s is the shearing rigidity of the member.

Twisting Moment: The strain energy stored in the member due to twisting moment is expressed by

$$U = \int_0^L \frac{T^2}{2GJ} dx \quad \textbf{(4.21)}$$

where *T* is the twisting moment; and *GJ* is the torsional rigidity of the member.

Example 4.18 Find the horizontal deflection at joint *C* of the pin-jointed frame as shown in Figure 4.26(a). *AE* is constant for all members.

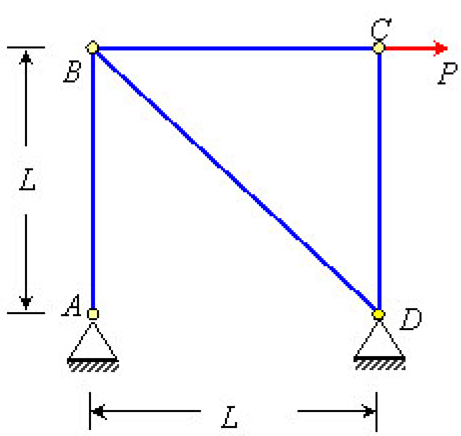

Figure 4.26(a)

Solution: The force in various members of the frame is shown in Figure 4.26(b). Calculation of strain energy of the frame is shown in Table 4.4.

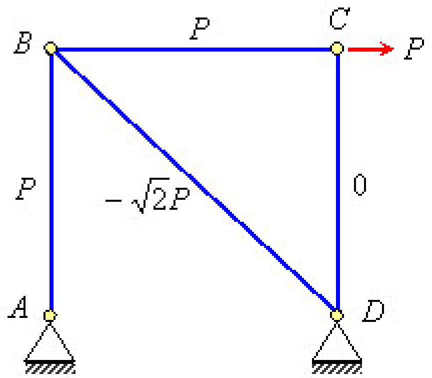

Figure 4.26(b). Forces due to applied load

Table 4.4

Member	**Length (*L*)**	**Force (*P*)**	$U = \frac{P^2 L}{2AE}$
AB	*L*	*P*	$P^2 L / 2AE$
BC	*L*	*P*	$P^2 L / 2AE$
BD	$\sqrt{2}L$	$-P\sqrt{2}$	$\sqrt{2}\ P^2 L / 2AE$
CD	*L*	0	0

$$\sum \quad (\sqrt{2}+1)P^2 L / AE$$

Horizontal displacement of joint *C* , $\Delta_{CH} = \frac{\partial U}{\partial P} = \frac{\partial}{\partial P}\left(\frac{(\sqrt{2}+1)P^2 L}{AE}\right)$

$$= \frac{2(\sqrt{2}+1)PL}{AE}(\rightarrow)$$

Example 4.19 A bar of uniform cross-section is bent into a quadrant of circle of radius *R* . One end of the bent is fixed and other is free. At the free end it carries a vertical load *W* . Determine the vertical and horizontal deflection at *A* .

Solution:

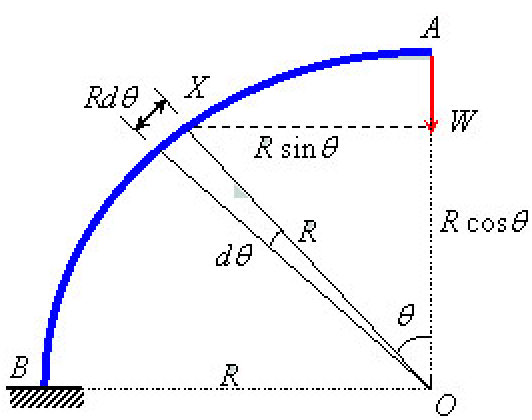

Figure 4.27(a)

***Vertical displacement of A* : The vertical displacement of *A* is given by**

$$\Delta_{AV} = \frac{\partial U}{\partial W}$$

For evaluation of the total strain energy in the system, consider a small element $Rd\theta$ as shown in the Figure. The bending moment at this element, $M_\theta = -W\,R\sin\theta$. Thus,

$$\Delta_{AV} = \frac{\partial}{\partial W}\left(\int_0^{\pi/2} \frac{M_\theta^2 R d\theta}{2EI}\right)$$

$$= \frac{1}{EI}\int_0^{\pi/2} M_\theta\left(\frac{\partial M_\theta}{\partial W}\right) R d\theta$$

$$= \frac{1}{EI}\int_0^{\pi/2} WR^3 \sin^2\theta\, d\theta$$

$$= \frac{WR^3}{EI}\int_0^{\pi/2}\left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \frac{\pi W R^3}{4EI}$$

Since there is no horizontal force acting at point *A* , apply a horizontal force, *F* at *A* as shown in Figure 4.27(b). From the Castigliano's theorem, the horizontal displacement of *A* due to applied external load *W* is given by

$$\Delta_{AH} = \left.\frac{\partial U}{\partial F}\right|_{F=0}$$

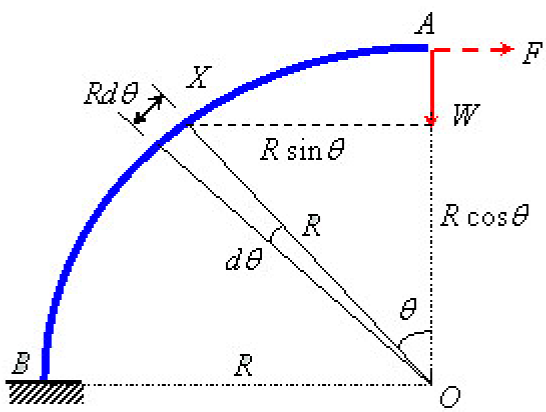

Figure 4.27(b)

The bending moment at the small element $Rd\theta$ is $M_\theta = -WR\sin\theta - FR(1-\cos\theta)$. Thus, the horizontal displacement of *A*

$$\Delta_{AH} = \left.\frac{\partial}{\partial F}\left(\int_0^{\pi/2} \frac{M_\theta^2 R d\theta}{2EI}\right)\right|_{F=0}$$

$$= \left.\frac{1}{EI}\int_0^{\pi/2} M_\theta\left(\frac{\partial M_\theta}{\partial F}\right) R\, d\theta\right|_{F=0}$$

$$= \frac{1}{EI}\int_0^{\pi/2}\left(-WR\sin\theta - FR(1-\cos\theta)\right)\left(\frac{\partial\left(WR\sin\theta - FR(1-\cos\theta)\right)}{\partial F}\right)R\,d\theta\Bigg|_{F=0}$$

$$= \frac{1}{EI}\int_0^{\pi/2}\left(-WR\sin\theta\right)\left(-R(1-\cos\theta)\right)R\,d\theta$$

$$= \frac{1}{EI}\int_0^{\pi/2}\left[-WR^3\left(-\sin\theta + \frac{\sin 2\theta}{2}\right)\right]d\theta$$

$$= -\frac{WR^3}{2EI}$$ (i.e. deflection is in $\leftarrow$ direction)

Example 4.20 Determine the deflection of the end *A* of the beam as shown in Figure 4.28. The flexibility of the spring is $f = L^3 / EI$.

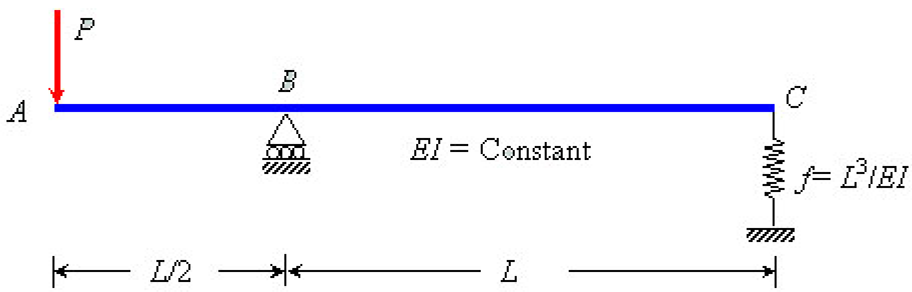

Figure 4.28

Solution: Reactions at support *B* and *C* are
$R_B = 1.5P$ (upward) and $R_C = 0.5P$ (downward)
Force in the spring = Reaction, $R_C = 0.5P$

Deflection under the load is given by

$$\Delta_A = \frac{\partial U}{\partial P}$$

where $U = U_{AB} + U_{BC} + U_S$ is the total strain energy stored in the system; U_{AB} is the energy stored in the member *AB* ; U_{BC} is the energy stored in the member *BC* ; and U_S = strain energy in the spring.

Strain energy in the spring is given by

$$U_S = \frac{1}{2} f \left(\frac{P}{2}\right)^2 = \frac{P^2L^3}{8EI}$$

Consider member *AB* : (*x* measured from *A*)

$$M_x = -Px$$

$$U_{AB} = \int_0^{L/2} \frac{M_x^2 dx}{2EI} = \int_0^{L/2} \frac{P^2 x^2 dx}{2EI} = \frac{P^2L^3}{48EI}$$

Consider member *BC* : (*x* measured from *C*)

$$M_x = -\frac{P}{2}x$$

$$U_{BC} = \int_0^L \frac{M_x^2 dx}{2EI} = \int_0^L \frac{P^2 x^2 dx}{8EI} = \frac{P^2 L^3}{24EI}$$

Thus,

$$U = U_{AB} + U_{BC} + U_S$$

$$= \frac{P^2 L^3}{8EI} + \frac{P^2 L^3}{48EI} + \frac{P^2 L^3}{24EI} = \frac{3P^2 L^3}{16EI}$$

The deflection of point *A*,

$$\Delta_A = \frac{\partial U}{\partial P} = \frac{3PL^3}{8EI}$$

Recap

In this course you have learnt the following

- Deflection by strain energy method.
- Evaluation of strain energy in member under different loading.
- Application of strain energy method for different types of structure.