Module 4 : Deflection of Structures

Lecture 3 : Principle of Virtual Work

Objectives

In this course you will learn the following

- Computation of deflection using principle of virtual work (PVW).
- Application to pin-jointed structure.
- Application of PVW to beams and frames.
- Simplified PVW for beams and frames using multiplication of bending moment diagram.

4.4 Principle of Virtual Work

Consider a structural system subjected to a set of forces (P_1, P_2, P_3... referred as *P* force) under stable equilibrium condition as shown in Figure 4.11(a). Further, consider a small element within the structural system and stresses on the surfaces caused by the *P* forces are shown in Figure 4.11(b) and referred as σ_P.

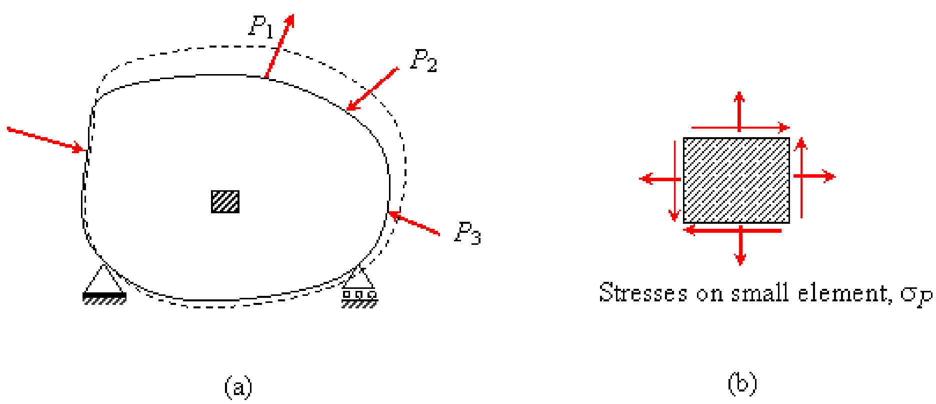


Figure 4.11 Deformable body subjected to external force

Let the body undergoes to a set of compatible virtual displacement δD . These displacements are imaginary and fictitious as shown by dotted line. While the body is displaced, the real forces acting on the body move through these displacements. These forces and virtual displacements must satisfy the principle of conservation of energy i.e.

$$\delta W_e = \delta W_i \qquad \textbf{(4.8)}$$

$$\sum_{i=1}^{n} P_i(\delta D)_i = \int_V \sigma_P(\delta\varepsilon)dV \qquad \textbf{(4.9)}$$

This is the principle of virtual work

If a system in equilibrium under a system of forces undergoes a deformation, the work done by the external forces (*P*) equals the work done by the internal stresses due to those forces, (σ_P).

In order to use the above principle for practical applications, we have to interchange the role of the forces and displacement. Let the structure acted upon by a virtual force is subjected to real displacements then the Eq. (4.9) can be written as

$$\sum_{i=1}^{n} D_i(\delta P)_i = \int_V \varepsilon(\delta\sigma_P)dV \qquad \textbf{(4.10)}$$

This is the principle of complimentary virtual work and used for computing displacements.

Consider a structure shown in Figure 4.12(a) and subjected to *P* force and it is required to find the displacement of point *C* in the direction specified. First apply a virtual force at *C* in the required

δF

direction. Next apply the external (real) loads acting on the structures as shown in Figure 4.12(a) with the virtual force remain in the position. The displacement of *C* in the required diection be Δ and the internal elements deform by an amount ΔL . Using Eq. (4.10)

$$\delta F \times \Delta = \sum \delta f \times \Delta L \tag{4.11}$$

The left hand side of Eq. (4.11) denotes the external work done by the virtual force δF moving through the real dispolacement Δ. On the other hand, the right hand side of Eq. (4.11) represents the internal work done by the virtual internal element forces d *f* moving through the displacement ΔL .

Since δF is arbitrary and for convenience let δF =1 (i.e. unit load). The Eq. (4.11) can be re-written as

$$1 \times \Delta = \sum f \times \Delta L \tag{4.12}$$

where *f* denotes the internal force in the members due to virtual unit load.

The right hand side of Eq. (4.12) will directly provide the displacement of point *C* due to applied external forces. This method is also known as unit load method.

Similarly for finding out a rotation, θ at any point of a loaded structure, the corresponding Eq. (4.12) will take place as

$$1 \times \theta = \sum f_\theta \times \Delta L \tag{4.13}$$

where f_θ denotes the internal force in the members due to virtual unit moment applied in the direction of interested θ.

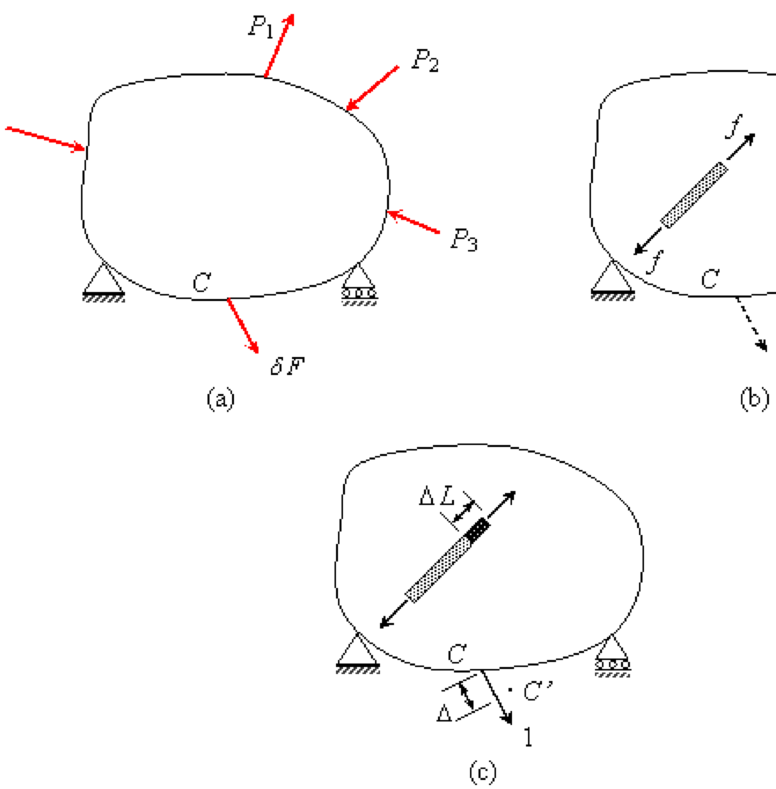


Figure 4.12

4.4.1 Application to Pin-Jointed Structures

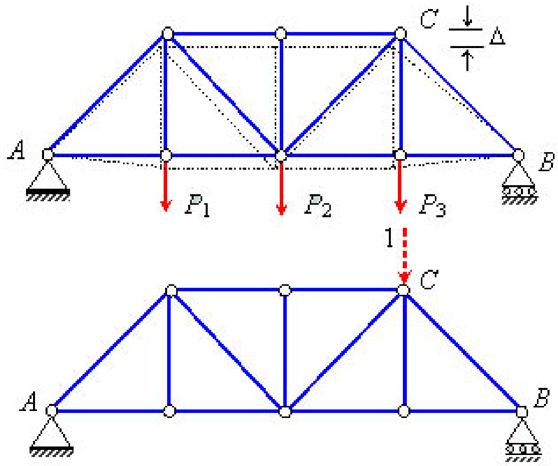


Figure 4.13

Consider a pin-jointed structure as shown in Figure 4.13 and subjected to external force P_1, P_2 and P_3. Let the vertical displacement of point *C*, Δ_{CV} is required. Under the action of the real external load, let the axial force in typical member be F_P and therefore, the deformation of the member $\Delta L = F_P L / AE$ (*L* and *AE* are the length and axial rigidity of typical member).

Apply a unit vertical load at *C* and substituting in Eq. (4.12) leads to

$$1 \times \Delta_{CV} = \sum F_u \times \Delta L$$

$$\Delta_{CV} = \sum F_u \frac{F_p L}{AE} \qquad \textbf{(4.14)}$$

The basic steps to be followed for finding the displacements of the pin-jointed structure are

1. Compute the axial force in various members (i.e. F_P) due to applied external forces.
2. Compute the axial force in various members (i.e. F_u) due to unit load applied in the direction of required displacement of the point.
3. Compute the product $F_u \dfrac{F_p L}{AE}$ for all members.
4. The summation $\sum F_u \dfrac{F_p L}{AE}$ will provide the desired displacement.
5. The axial force shall be taken as positive if tensile and negative if compressive.
6. The positive $\sum F_u \dfrac{F_p L}{AE}$ implies that the desired displacement is in the direction of applied unit load and negative quantity will indicate that the desired displacement is in the opposite direction of the applied unit laod.

Example 4.9 Find the horizontal and vertical deflection at joint *C* of the pin-jointed frame shown in Figure 4.14. *AE* is constant for all members.

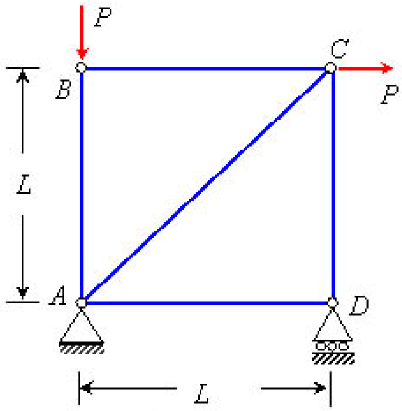
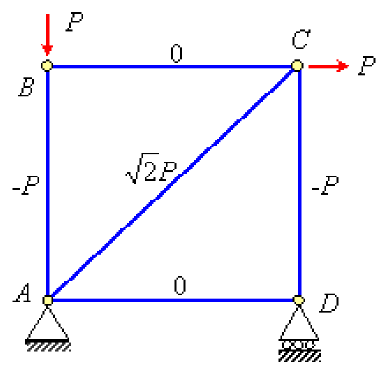
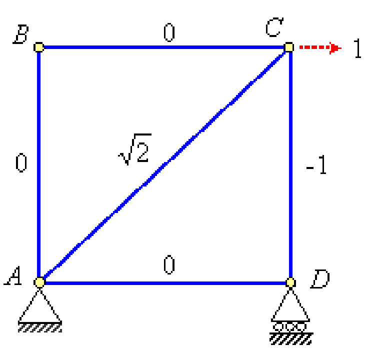


Figure 4.14

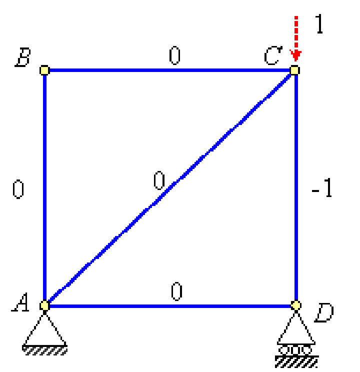
Solution: Calculate F_p forces i.e. force in various members of the truss due to the applied loading. These can be obtained by considering the equilibrium of various joints as marked in Figure 4.14(b).



Forces due to applied loads (F_p)



Forces due to unit load at C (F_u)



Forces due to unit load at C (F_u)

Table 4.2

Member	Length	F_p	For δ_{CH}		For δ_{CV}	
			F_u	$F_p\ F_u$ L	F_u	$F_u\ F_p$ L
AB	L	-P	0	0	0	0
BC	L	0	0	0	0	0
CD	L	-P	-1	PL	-1	PL
DA	L	0	0	0	0	0
AC	$\sqrt{2}L$	$P\sqrt{2}$	$\sqrt{2}$	$2\sqrt{2}PL$	0	0
			$\sum$	$(2\sqrt{2}+1)$		PL

The computation of $\sum F_u \frac{F_p L}{AE}$ for two desired displacements of pin-jointed frame are shown in Table 4.2.

Horizontal displacement of joint *C* , $\delta_{CH} = \frac{(2\sqrt{2}+1)PL}{AE}$ $(\longrightarrow)$

Vertical displacement of joint *C* , $\delta_{CV} = \frac{PL}{AE}$ $(\downarrow)$

Example 4.10 For the pin-jointed structure shown in the Figure 4.15, find the horizontal and vertical displacement of the joint *D* . The area of cross-section, *A* =500 mm^2 and *E* =200,000 N/mm^2 for all the members.

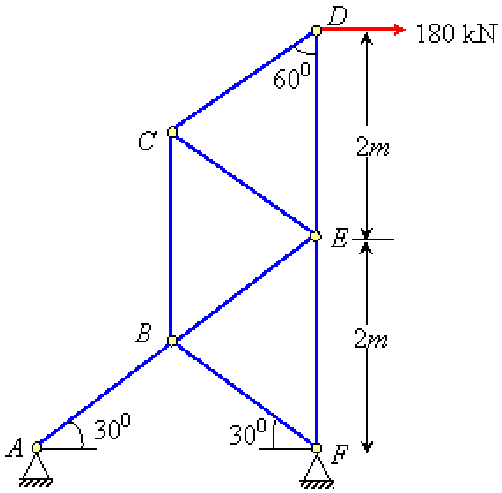


Figure 4.15

Solution: The axial rigidity of the members, $AE = 500 \times 2 \times 10^5 \times 10^3 = 10^5$ kN. The computation of the desired displacements is presented in Table 4.3

Table 4.3

Member	Length L(m)	F_p	For δ_{DH}		For δ_{DV}	
			F_u	F_p F_u L	F_u	F_u F_p L
AB	2	$720/\sqrt{3}$	$4/\sqrt{3}$	1920	0	0
BC	2	$360/\sqrt{3}$	$2/\sqrt{3}$	480	0	0
CD	2	$360/\sqrt{3}$	$2/\sqrt{3}$	480	0	0
DE	2	$-180/\sqrt{3}$	$-1/\sqrt{3}$	120	1	$-360/\sqrt{3}$
EF	2	$-180/\sqrt{3}$	$-\sqrt{3}$	1080	1	$-360/\sqrt{3}$
CE	2	$-360/\sqrt{3}$	$-2/\sqrt{3}$	480	0	0
BF	2	$360/\sqrt{3}$	$2/\sqrt{3}$	480	0	0
			Σ	5520		$-1440/\sqrt{3}$

The horizontal deflection of $D = \frac{5520}{10^5} \times 10^3$ = 55.2 mm $(\longrightarrow)$

The vertical deflection of $D = -\frac{1440/\sqrt{3}}{10^5} \times 10^3$ = - 8.31 mm $(\uparrow)$ = 8.31 mm $(\downarrow)$

4.4.2 Application to beams and frames

In order to find out the vertical displacement of *C* of the beam shown in Figure 4.16(a), apply a unit load as shown in Figure 4.16(b).

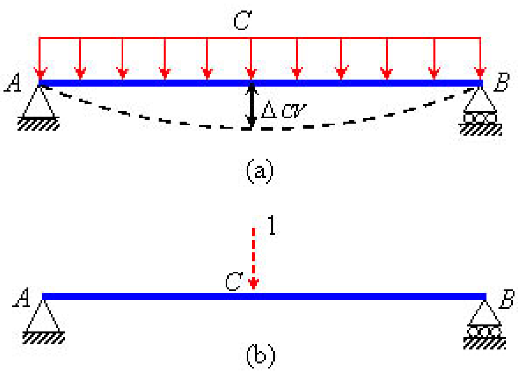


Figure 4.16

The internal virtual work is considered mainly due to bending and caused due to internal moments M_u under going the rotation $d\theta$ due to the applied loading. (internal virtual work done by shearing forces and axial forces is small in comparison to the bending moments and hence ignored). Since the $d\theta = M_p dx/EI$ where M_p is the moment due to applied loading, the Eq. (4.12) for the displacement of *C* will take a shape of

$$1 \times \Delta_{CV} = \int_0^L M_u \times d\theta$$

$$\Delta_{CV} = \int_0^L M_u \frac{M_p dx}{EI} \qquad \textbf{(4.15)}$$

The basic steps to be followed for finding the displacement or slope of a beams and frames are summarized as

1. Compute the bending moment (i.e. M_p) due to applied external forces.
2. Compute the bending moment (i.e. M_u) due to unit load applied in the direction of required displacement or slope.
3. Compute the integral $\int_0^L M_u \frac{M_p dx}{EI}$ over the entire members of the beam or frame which will provide the desired displacement.
4. The bending moment shall be taken as positive if sagging and negative if hogging (in case of beams).
5. The positive $\int_0^L M_u \frac{M_p dx}{EI}$ implies that the desired displacement is in the direction of applied unit load and negative quantity will indicate that the desired displacement is in the opposite direction of the applied unit load.

Example 4.11 Determine the slope and deflection of point *A* of the cantilever beam *AB* with length *L* and constant flexural rigidity *EI.*

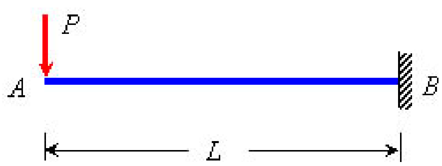


Figure 4.17(a)

Solution: Deflection under the Load - Apply a vertical unit load at point *A* of the beam as shown in Figure 4.17(b). Consider any point *X* at a distance of *x* from *A* ,

$$M_p = -P \times x = -Px$$

$$M_u = -1 \times x = -x$$

The vertical deflection of point A is given by

$$\Delta_{AV} = \int_0^L M_u \frac{M_p dx}{EI}$$

$$= \int_0^L (-x) \frac{(-Px)dx}{EI}$$

$$\Delta_{AV} = \frac{PL^3}{3EI}$$

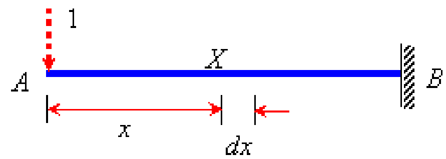


Figure 4.17(b)

Slope at the free end: Apply a unit couple at point *A* of the beam as shown in Figure 4.17(c). Consider any point *X* at a distance of *x* from *A* .

$$M_p = -P \times x = -Px$$

$$M_u = -1$$

The slope at *A* is given by

$$= \int_0^L (-1) \frac{(-Px)\,dx}{EI}$$

$$\theta_A = \frac{PL^2}{2EI}$$

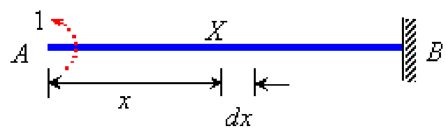


Figure 4.17(c)

Example 4.12. Determine mid-span deflection and end slopes of a simply supported beam of span *L* carrying a udl *w* per unit length.

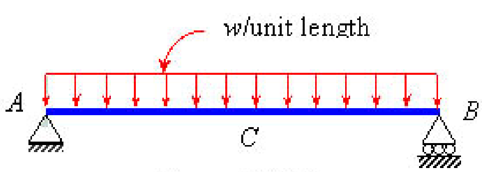


Figure 4.18(a)

Solution: *Mid-span deflection* **:** Apply a unit load at mid span as shown in Figure 4.18(b). Consider any point *X* at a distance of *x* from *A*

$$M_p = \frac{wL}{2} \times x - \frac{wx^2}{2}$$

$$= \frac{wx(L-x)}{2} \qquad (0 < x < L)$$

$$M_u = \frac{1}{2} \times x = \frac{x}{2} \qquad (0 < x < L/2)$$

$$M_u = \frac{1}{2} \times x - 1 \times \left(x - \frac{L}{2}\right) = \frac{L-x}{2} \qquad (L/2 < x < L)$$

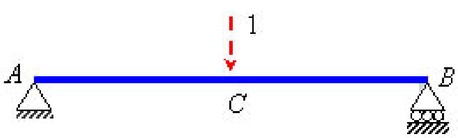


Figure 4.18(b)

The vertical deflection of point *C* is given by

$$\Delta_{CV} = \int_0^L M_u \frac{M_p dx}{EI}$$

$$= \int_0^{L/2} \left(\frac{x}{2}\right) \frac{\left(\frac{wx(L-x)}{2}\right)dx}{EI} + \int_{L/2}^{L} \left(\frac{L-x}{2}\right) \frac{\left(\frac{wx(L-x)}{2}\right)dx}{EI}$$

$$\delta_{CV} = \frac{5wL^4}{384\,EI}$$

End slopes **:** Applying a unit couple at *A* as shown in Figure 4.18(c). Consider any point *X* at a distance of *x* from *A*

$$M_p = \frac{wL}{2} \times x - \frac{wx^2}{2}$$

$$= \frac{wx(L-x)}{2} \qquad (0 < x < L)$$

$$M_u = 1 - \frac{1}{L} \times x = \frac{L-x}{L} \quad (0 < x < L/2)$$

Figure 4.18(c)

The slope at *A* is given by

$$\theta_A = \int_0^L M_u \frac{M_p dx}{EI}$$

$$= \int_0^L \left(\frac{L-x}{L}\right) \frac{\left(\frac{wx(L-x)}{2}\right) dx}{EI}$$

$$\Theta_A = \frac{wL^3}{24\,EI}$$

Due to symmetry $\theta_C = \theta_A = \dfrac{wL^3}{24EI}$ (anti-clockwise direction)

Example 4.13 Determine vertical deflection and rotation of point *B* of the beam shown in Figure 4.19(a). The beam is subjected to a couple M_0 at *C* .

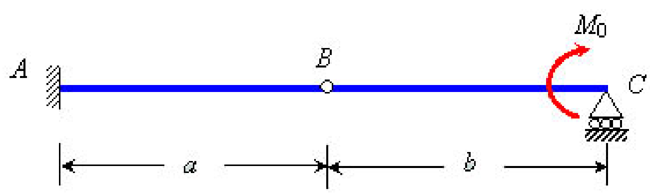


Figure 4.19(a)

Solution: Vertical deflection of B **:** Apply a unit load at *B* as shown in Figure 4.19(b). Consider any point *X* at a distance of *x* from *C*

$$M_p = \frac{M_0}{b} \times x - M_0$$

$$= \frac{M_0(x-b)}{b} \qquad (0 < x < a + b)$$

$$M_u = 0 \qquad (0 < x < b)$$

$$M_u = -1 \times (x-b) = b - x \qquad (b < x < a + b)$$

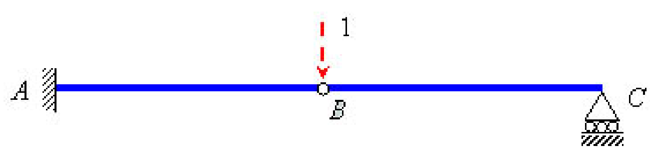


Figure 4.19(b)

The vertical deflection of point *B* is given by

$$\Delta_{BV} = \int_0^{a+b} M_u \frac{M_p dx}{EI}$$

$$= \int_0^a (0)\frac{\left(\frac{M_0(x-b)}{b}\right)dx}{EI} + \int_b^{a+b} (b-x)\frac{\left(\frac{M_0(x-b)}{b}\right)dx}{EI}$$

$$\Delta_{BV} = -\frac{M_0 a^3}{3EIb} \quad \text{(i.e. in the upward direction)}$$

Rotation of B **:** Apply a unit couple at *B* as shown in Figure 4.19(c). Consider any point *X* at a distance of *x* from *C*

$$M_p = \frac{M_0(x-b)}{b} \qquad (0 < x < a+b)$$

$$M_u = \frac{1}{b}\times x = \frac{x}{b} \qquad (0 < x < a+b)$$

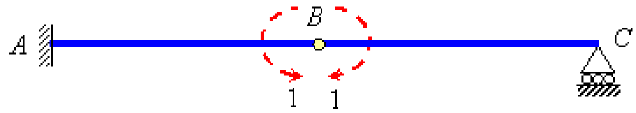


Figure 4.19(c)

The rotation of point *B* is given by

$$\theta_B = \int_0^{a+b} M_u \frac{M_p dx}{EI}$$

$$= \int_0^{a+b} \left(\frac{x}{b}\right)\frac{\left(\frac{M_0(x-b)}{b}\right)dx}{EI}$$

$$\theta_B = \frac{M_0(a+b)^2(2a-b)}{6EIb^2}$$

Example 4.14. Determine horizontal deflection of *C* and slope at *A* of a rigid-jointed plane frame as shown in Figure 4.20(a). Both members of the frame have same flexural rigidity, *EI* .

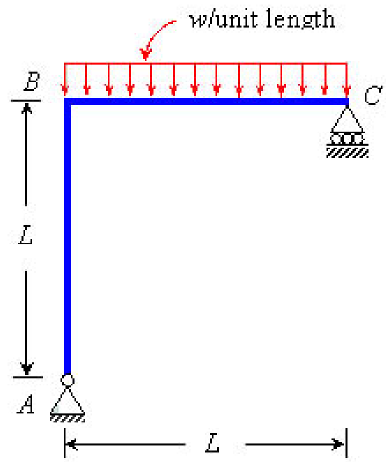


Figure 4.20(a)

Solution: *Horizontal deflection of C* **:** Apply a unit load *C* as shown in Figure 4.20(b).

Consider *AB* : (*x* measured *A*)

$$M_p = 0$$

$$M_u = 1 \times x = x$$

Consider *BC* : (*x* measured *C*)

$$M_p = \frac{wL}{2} \times x - \frac{wx^2}{2}$$
$$= \frac{wx(L-x)}{2}$$
$$M_u = 1 \times x = x$$

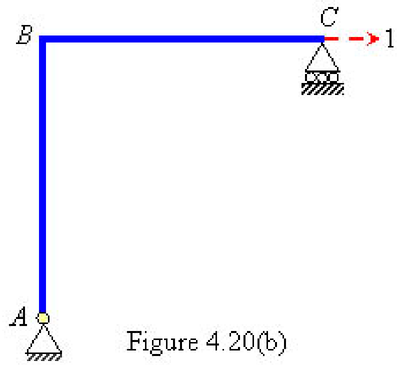


Figure 4.20(b)

The horizontal deflection of point *C* is given by

$$\Delta_{CH} = \int_{AB} M_u \frac{M_p dx}{EI} + \int_{BC} M_u \frac{M_p dx}{EI}$$

$$= \int_0^L (x) \frac{(0) dx}{EI} + \int_0^L (x) \frac{\left(\frac{wx(L-x)}{2}\right) dx}{EI}$$

$$\delta_{CH} = \frac{wL^4}{24 EI}$$

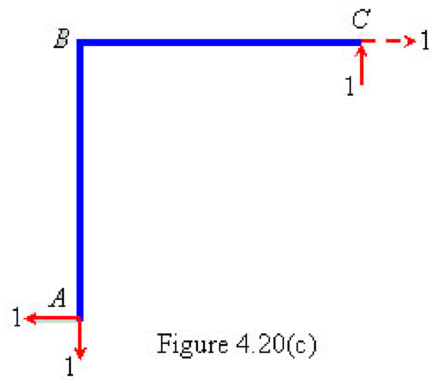


Figure 4.20(c)

Rotation at A : Applying a couple at *A* as shown in Figure 4.20(d).

Consider *AB* : (*x* measured *A*)

$$M_p = 0$$

$$M_u = 1$$

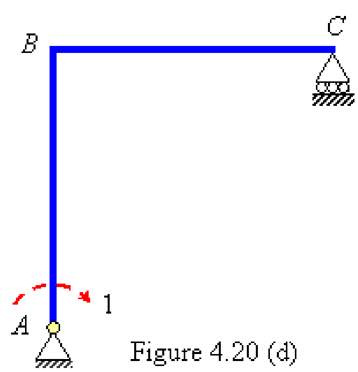


Figure 4.20 (d)

Consider *BC* : (*x* measured *C*)

$$M_p = \frac{wL}{2} \times x - \frac{wx^2}{2}$$
$$= \frac{wx(L-x)}{2}$$
$$M_u = \frac{1}{L} \times x = \frac{x}{L}$$

The slope at *A* is given by

$$\theta_A = \int_{AB} M_u \frac{M_p dx}{EI} + \int_{BC} M_u \frac{M_p dx}{EI}$$

$$= 0 + \int_0^L \left(\frac{x}{L}\right) \frac{\left(\frac{wx(L-x)}{2}\right) dx}{EI}$$

$$\theta_A = \frac{wL^3}{24 EI}$$

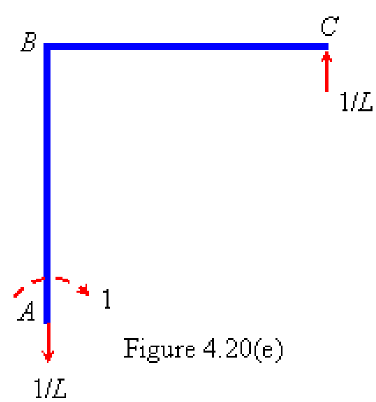


Figure 4.20(e)

4.4.3 Moment diagrams multiplication method for beams and frames

Recall the Eq. (4.15) in which the bending deflection of the beams and frames are obtained by the integration of the two bending moments variations (i.e. M_P and M_u) over a length of the members. However, for a uniform beam section (i.e. *EI* is constant) such integrals can be readily derived depending upon the various shapes of the bending moment diagrams. The computation of integral $\int_0^L M_u M_p \, dx$ is given in the Table 4.A1. The various steps for this method for finding deflections of the beams and frame are:

1. Draw the bending moment diagram of given beam or frame due to applied external loading (i.e. M_P diagram).
2. Draw the corresponding bending moment diagram due to unit load applied in the direction of interested deflection (i.e. M_u diagram).
3. Compute the desired deflection by computing the $\int_0^L M_u M_p \, dx$ with the help of results shown in **Table 4.A1**.

Example 4.15 Determine the deflection under the load and point *D* of a simply supported beam with overhang as shown in Figure 4.21

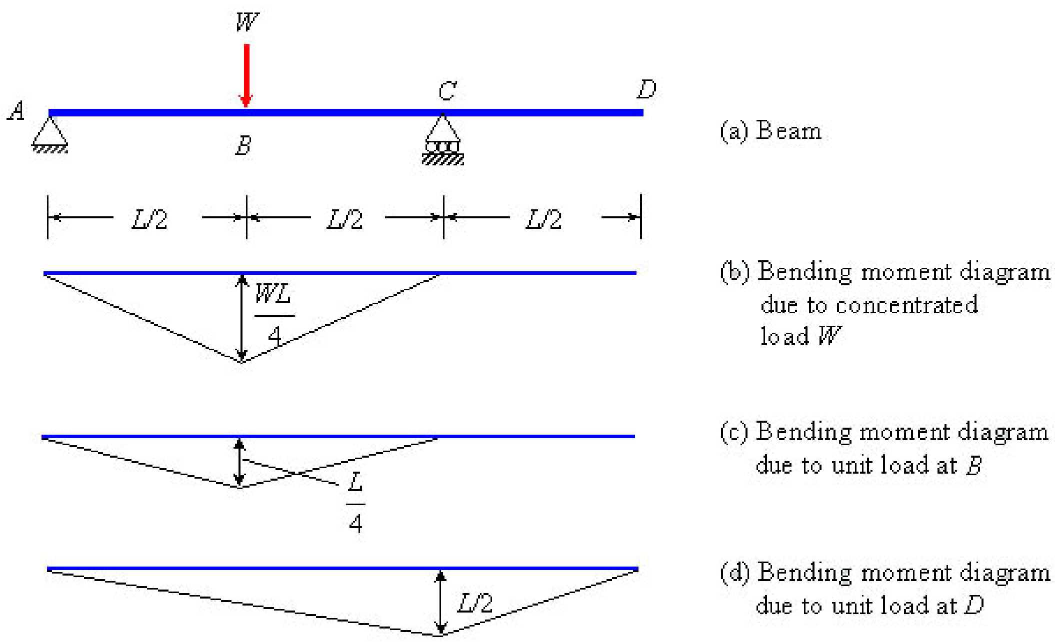


Figure 4.21

Solution: Bending moment diagram (i.e. M_P diagram) due to concentrated load *W* is shown in Figure 4.21(b).

Deflection under the Load **:** Apply a vertical unit load in place *of W* . The bending moment diagram due to this load is shown in Figure 4.21(c). The vertical deflection under the load is obtained by multiplying the bending moment diagrams of Figure 4.21(b) and (c) and is given by

$$\Delta_B = \frac{1}{EI}\int_{AB} M_u M_p dx + \frac{1}{EI}\int_{BC} M_u M_p dx + \frac{1}{EI}\int_{CD} M_u M_p dx$$

$$= \frac{1}{EI}\left(\frac{1}{3}\times\frac{WL}{4}\times\frac{L}{4}\times\frac{L}{2}\right) + \frac{1}{EI}\left(\frac{1}{3}\times\frac{WL}{4}\times\frac{L}{4}\times\frac{L}{2}\right) + 0 \quad \textbf{(refer Table 4.A1)}$$

$$= \frac{WL^3}{48EI}$$

Deflection of the free end **:** Apply a unit vertical load acting upward at point *D* of the beam. The bending moment diagram due to this load is shown in Figure 4.21(d). The vertical deflection under the load is obtained by multiplying the bending moments diagrams of Figure 4.21(b) and (d) and is given by

$$\Delta_D = \frac{1}{EI}\int_{AC} M_u M_p dx + \frac{1}{EI}\int_{CD} M_u M_p dx$$

$$= \frac{1}{EI}\left(\frac{1}{6}\times(1+\frac{1}{2})\times\frac{WL}{4}\times\frac{L}{2}\times L\right)+0$$ **(refer Table 4.A1)**

$$= \frac{WL^3}{32EI}$$

Example 4.16 Using the diagram multiplication method, determine the deflection under the load and end slopes of a non-prismatic simply supported beam.

Solution: Bending moment (B.M.) diagram (i.e. M_p diagram) due to concentrated load *W* on the beam is shown in Figure 4.22(b).

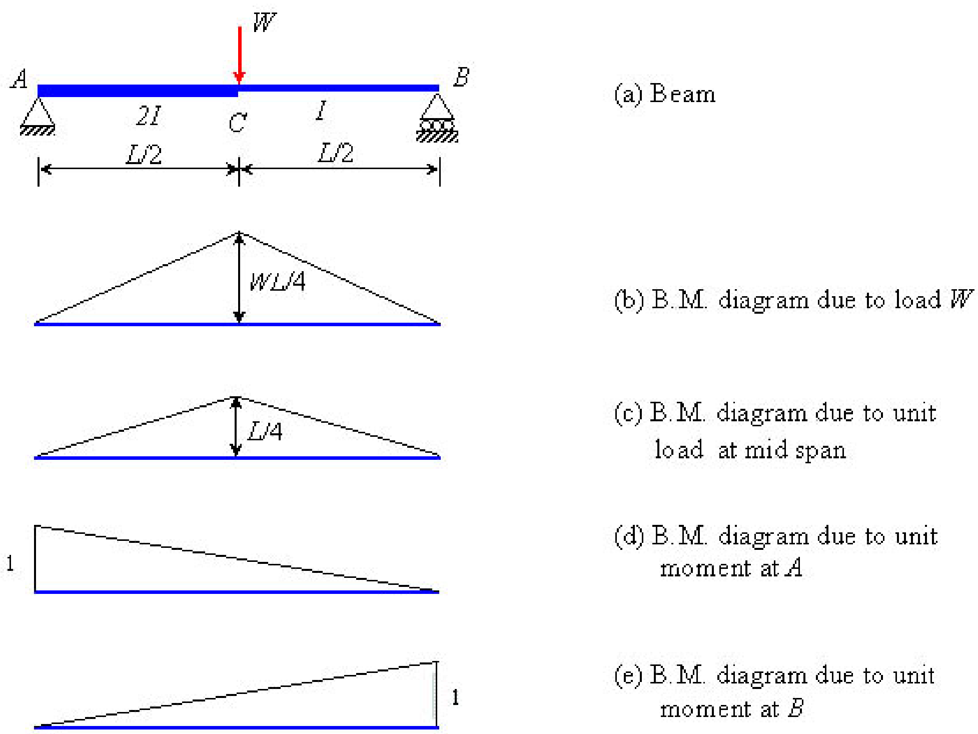


Figure 4.22

Mid-span deflection : Apply a unit load in the downward direction at *C* . Deflection at *C* is given by multiplying the diagrams of Figure 4.22 (b) and (c) as follows

$$\Delta_C = \frac{1}{2EI}\int_{AC} M_u M_p dx + \frac{1}{EI}\int_{CB} M_u M_p dx$$

$$= \left(\frac{1}{2EI}\right)\left[\frac{1}{3}\times\frac{WL}{4}\times\frac{L}{4}\times\frac{L}{2}\right]+\left(\frac{1}{EI}\right)\left[\frac{1}{3}\times\frac{WL}{4}\times\frac{L}{4}\times\frac{L}{2}\right]$$

$$= \frac{WL^3}{192EI}+\frac{WL^3}{96EI}=\frac{WL^3}{64EI}$$

Slope at A : Apply a unit couple at *A* acting in the clockwise direction and plot the bending moment diagram of the beam as shown in Figure 4.22(d). The slope at *A* is given by multiplying the diagrams of Figure 4.22 (b) and (d) as follows

$$\theta_A = \left(\frac{1}{2EI}\right)\left[\frac{1}{6}\times\frac{WL^4}{4}\times\left(2\times\frac{1}{2}+1\right)\times\frac{1}{2}\right]+\left(\frac{1}{EI}\right)\left[\frac{1}{3}\times\frac{1}{2}\times\frac{WL}{4}\times\frac{L}{2}\right]$$

$$= \left(\frac{1}{2EI}\right)\left[\frac{WL}{24} \times \frac{L}{2} \times 2\right] + \frac{WL^2}{48EI} = \frac{WL^2}{24EI} \text{ (clockwise direction)}$$

Slope at B : Apply a unit couple at *B* acting in the anti-clockwise direction and plot the bending moment diagram of the beam as shown in Figure 4.22(e). The slope at *B* is given by multiplying the diagrams of Figure 4.22 (b) and (e) as follows

$$\theta_B = \left(\frac{1}{2EI}\right)\left[\frac{1}{3} \times \frac{WL}{4} \times \frac{1}{2} \times \frac{L}{2}\right] + \left(\frac{1}{EI}\right)\left[\frac{1}{6} \times \frac{WL}{4} \times 2 \times \frac{L}{2}\right]$$

$$= \frac{WL^2}{96EI} + \frac{WL^2}{24EI} = \frac{5WL^2}{96EI}$$

Example 4.17 Using the diagram multiplication method, determine the horizontal displacement and rotaion of point *C* of the rigid-jointed plane frame shown in Figure 4.23. Both the members of the frame have same EI value.

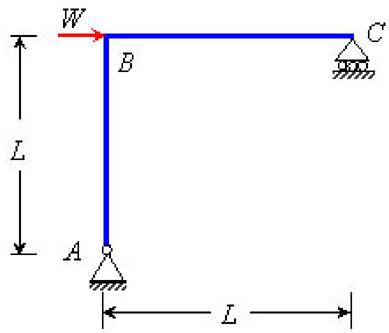
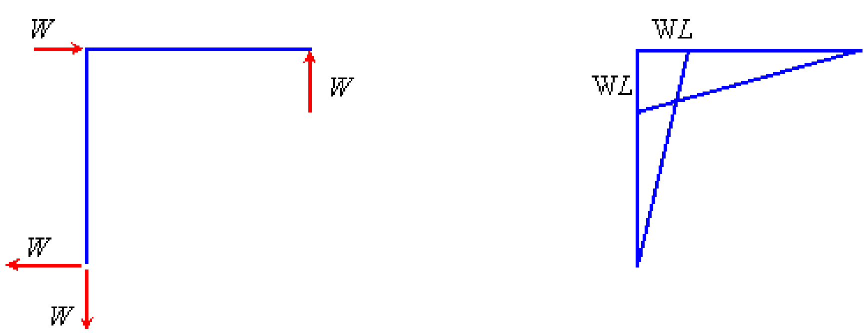


Figure 4.23(a)

Solution: The free-body and bending moment diagram (B.M.D.) of the frame due to applied loading are shown in Figures 4.23(b) and (c), respectively.



(b) Free body diagram

(c) B.M.D. due to applied load

Figure 4.23(b)-(c)

Horizontal deflection of C : Apply a horizontal force at *C* as shown in Figure 4.23(d) and plot the bending moment diagram as shown in Figure 4.23(e). The horizontal deflection at *C* is given by multiplying the diagrams of Figure 4.22 (c) and (e) as follows

$$\Delta_C = \frac{1}{EI}\int_{AB} M_u M_p dx + \frac{1}{EI}\int_{BC} M_u M_p dx$$

$$= \frac{1}{3EI} \times WL \times L \times L + \frac{1}{3EI} \times WL \times L \times L$$

$$= \frac{2WL^3}{3EI} \ (\rightarrow)$$

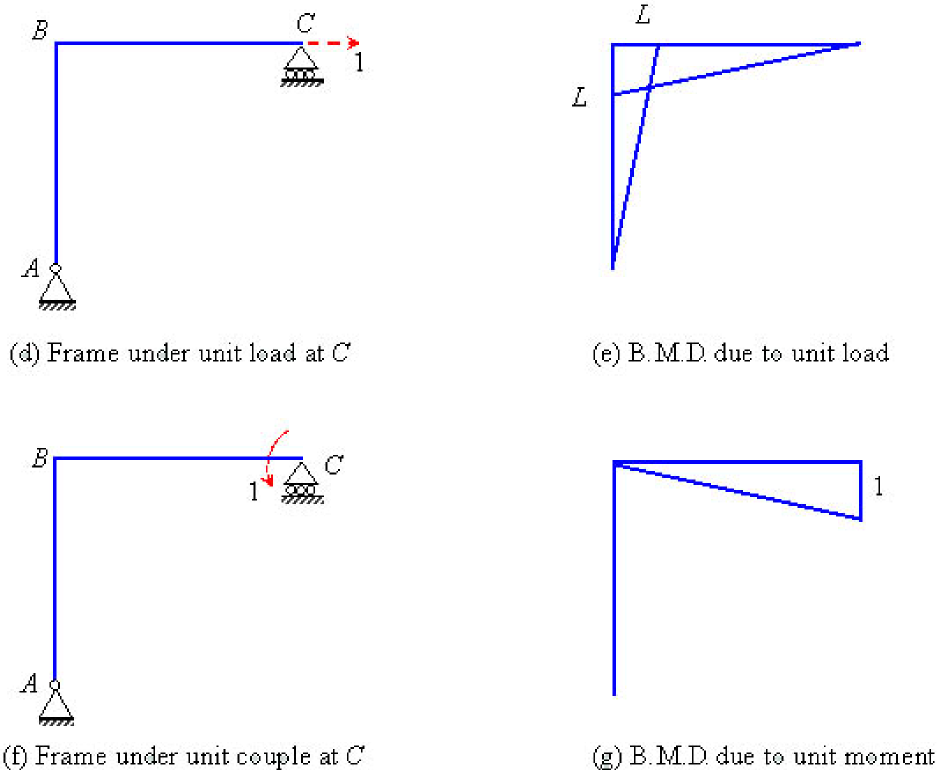


Figure 4.23(d)-(g)

Rotation of C : Apply a unit couple at *C* as shown in Figure 4.23(f) and plot the bending moment diagram as shown in Figure 4.23(g). The slope at *C* is given by multiplying the diagrams of Figure 4.22 (c) and (g) as follows

$$\theta_C = \frac{1}{EI}\int_{AB} M_u M_p dx + \frac{1}{EI}\int_{BC} M_u M_p dx$$

$$= 0 + \frac{1}{EI} \times \frac{1}{6} \times WL \times 1 \times L$$

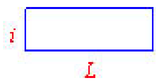
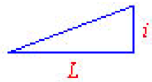
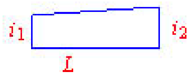
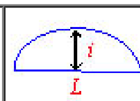
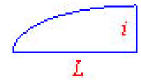
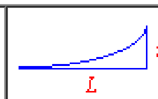
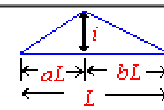
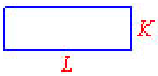
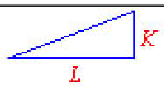
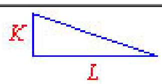
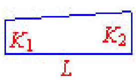
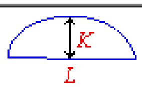
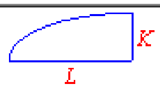
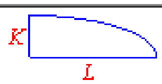
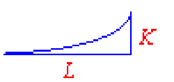
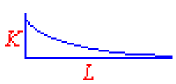
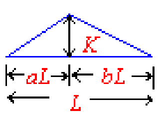
$$= \frac{WL^2}{6EI} \ \text{(anti-clockwise direction)}$$

Recap

In this course you have learnt the following

- Computation of deflection using principle of virtual work (PVW).
- Application to pin-jointed structure.
- Application of PVW to beams and frames.
- Simplified PVW for beams and frames using multiplication of bending moment diagram.

Table 4.A1 Evaluation of the integral $\int M_u M_p dx$

M_u \ M_P							
	KiL	1/2 *KiL*	1/2 *K* $(i_1 + i_2)L$	2/3 *KiL*	2/3 *KiL*	1/3 *KiL*	1/2 *KiL*
	1/2 *KiL*	1/3 *KiL*	1/6 *K* $(i_1 + 2i_2)L$	1/3 *KiL*	5/12 *KiL*	1/4 *KiL*	1/6(1 + *a*) *KiL*
	1/2 *KiL*	1/6 *KiL*	1/6 *K* $(2i_1 + i_2)L$	1/3 *KiL*	1/4 *KiL*	1/12 *KiL*	1/6(1 + *b*) *KiL*
	1/2 $(K_1 + K_2)$ *iL*	1/6 $(K_1 + 2K_2)$ *iL*	1/6 $(2K_1i_1 + K_1i_2 + K_2i_1 + 2K_2i_2)L$	1/3 $(K_1 + K_2)$ *iL*	1/12 $(3K_1 + 3K_2)$ *iL*	1/12 $(K_1 + 3K_2)$ *iL*	1/6((1 + *b*) K_1 + (1 + *a*) K_2)*iL*
	2/3 *KiL*	1/3 *KiL*	1/3 *K* $(i_1 + i_2)L$	8/15 *KiL*	7/15 *KiL*	1/5 *KiL*	1/3(1 + *ab*) *KiL*
	2/3 *KiL*	5/12 *KiL*	1/12 *K* $(3i_1 + 5i_2)L$	7/15 *KiL*	8/15 *KiL*	3/10 *KiL*	1/12(5 - *b* - b^2) *KiL*
	2/3 *KiL*	1/4 *KiL*	1/12 *K* $(5i_1 + 3i_2)L$	7/15 *KiL*	11/30 *KiL*	2/15 *KiL*	1/12(5 - *a* - a^2) *KiL*
	1/3 *KiL*	1/14 *KiL*	1/12 *K* $(i_1 + 3i_2)L$	1/5 *KiL*	3/10 *KiL*	1/5 *KiL*	1/12(1 + *a* + a^2) *KiL*
	1/3 *KiL*	1/12 *KiL*	1/12 *K* $(3i_1 + i_2)L$	1/5 *KiL*	2/15 *KiL*	1/30 *KiL*	1/12(1 + *b* + b^2) *KiL*
	1/2 *KiL*	1/6(1+ *a*) *KiL*	1/6 *KL* ((1 + *b*) i_1 + (1 + *a*) i_2)	1/3(1 + *ab*) *KiL*	1/12(5 - *b* - b^2) *KiL*	1/12(1 + *a* + a^2) *KiL*	1/3 *KiL*