Module 4 : Deflection of Structures

Lecture 2 : Conjugate Beam Method

Objectives

In this course you will learn the following

- Computation of deflection using conjugate beam method.

4.3 Conjugate Beam Method

The conjugate beam method is an extremely versatile method for computation of deflections in beams. The relationships between the loading, shear, and bending moments are given by

$$\frac{d^2M}{dx^2} = \frac{dV}{dx} = -w(x) \qquad \textbf{(4.7)}$$

where *M* is the bending moment; *V* is the shear; and *w* (*x*) is the intensity of distributed laod.

Similarly, we have the following

$$\frac{d^2v}{dx^2} = \frac{d\theta}{dx} = \frac{M}{EI} \qquad \textbf{(4.8)}$$

A comparison of two set of equations indicates that if *M* / *EI* is the loading on an imaginary beam, the resulting shear and moment in the beam are the slope and displacement of the real beam, respectively. The imaginary beam is called as the “ **conjugate beam** ” and has the same length as the original beam**.**

There are two major steps in the conjugate beam method. The first step is to set up an additional beam, called "**conjugate beam,**" and the second step is to determine the “ **shearing forces** ” and “ **bending moments** ” in the conjugate beam.

The **loading diagram** showing the *elastic loads* acting on the conjugate beam is simply the bending-moment diagram of the actual beam divided by the flexural rigidity *EI* of the actual beam. This *elastic load* is downward if the *bending moment* is sagging.

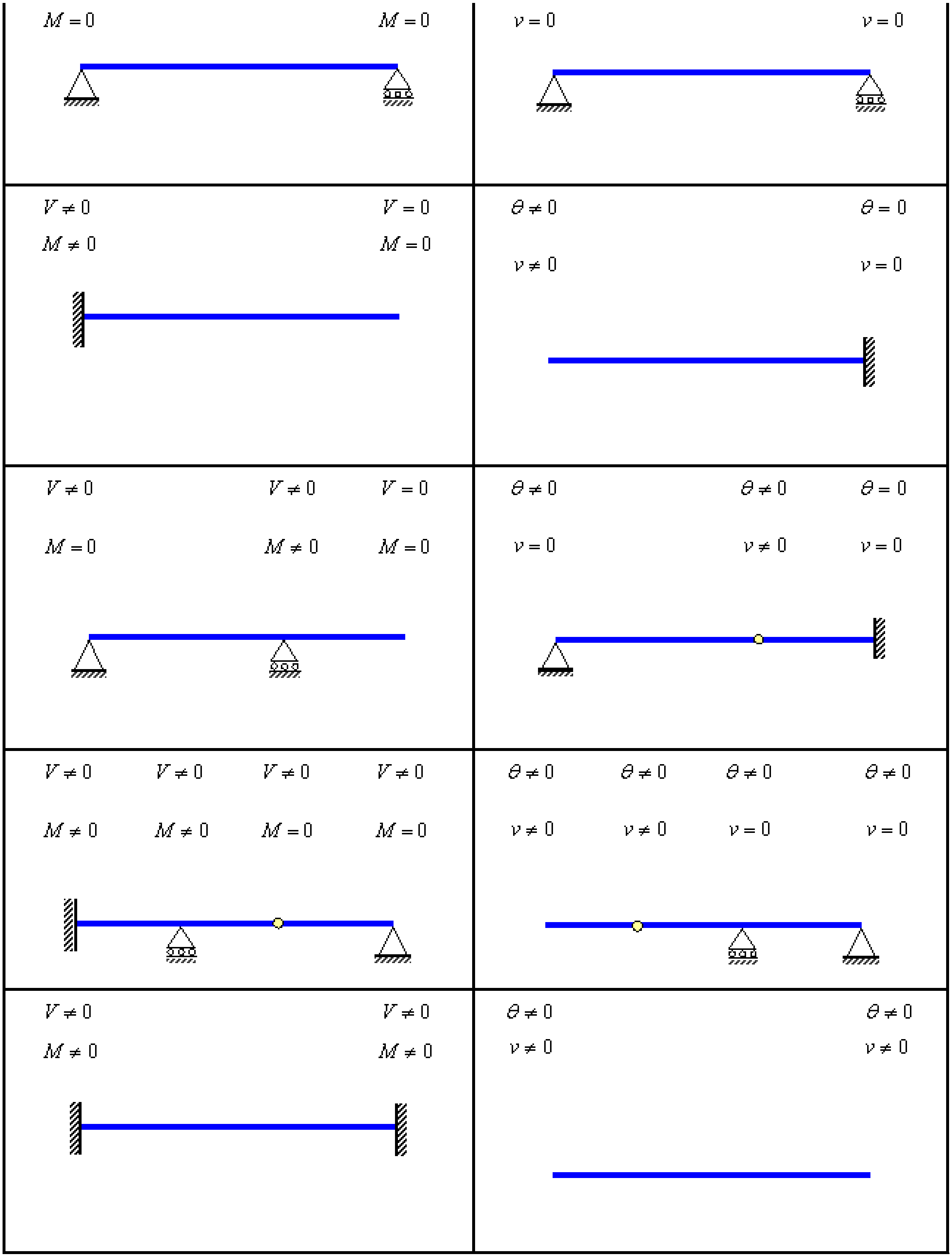
For each existing support condition of the actual beam, there is a *corresponding support condition* for the conjugate beam. Table 4.1 shows the corresponding conjugate beam of different types of actual beams. The actual beam as well as the conjugate beam are always in static **equilibrium condition** .

The slope of (the centerline of) the actual beam at any cross-section is equal to the “ **shearing force** ” at the corresponding cross-section of the conjugate beam. This slope is positive or anti-clockwise if the “ **shearing force** ” is positive — to rotate the beam element anti-clockwise — in beam convention . The deflection of (the centerline of) the actual beam at any point is equal to the “ **bending moment** ” of the conjugate beam at the corresponding point. This deflection is downward if the “ **bending moment** ” is positive — to cause top fiber in compression — in beam convention . The positive shearing force and bending moment are shown below in Figure 4.7.

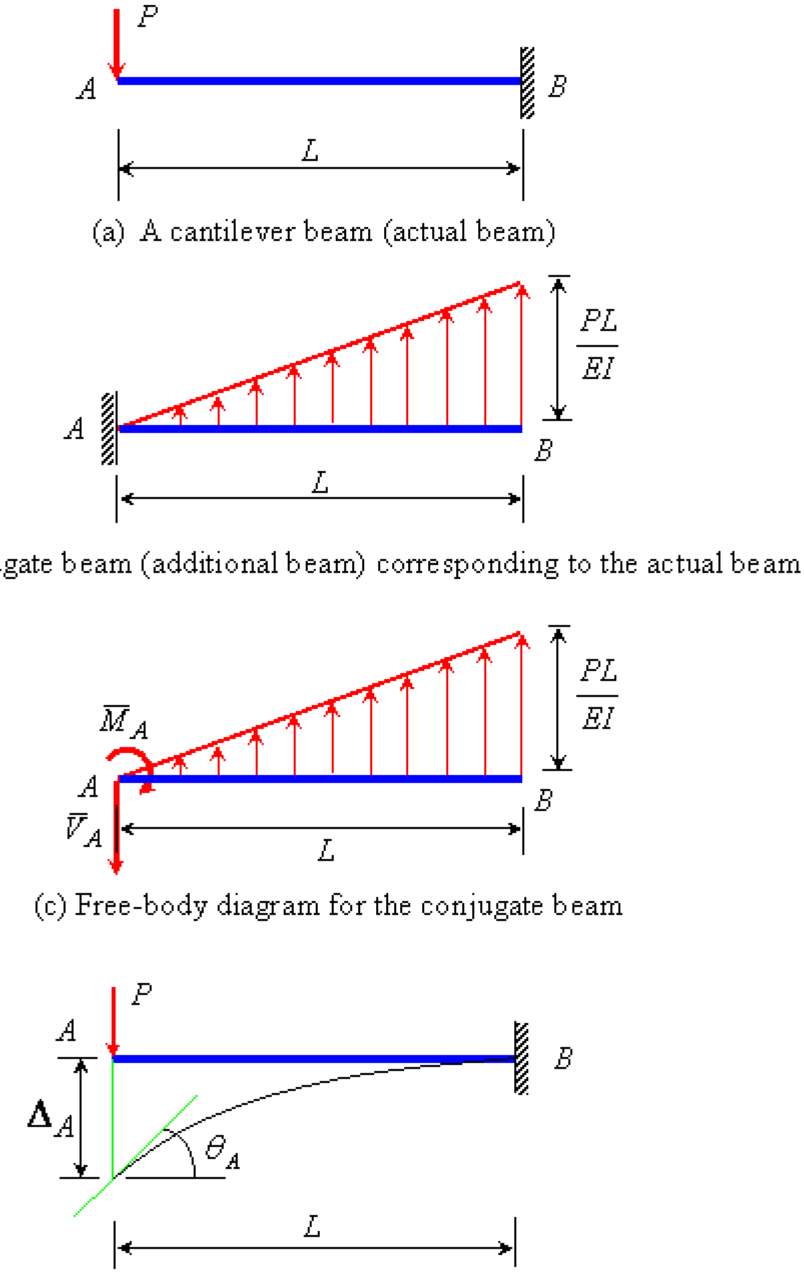
Figure 4.7 Positive shearing force and bending moment

Table 4.1 Real and Conjugate beams for different structures

REAL SRUCTURE	CONJUGATE STRUCTURE
$V \neq 0$ $\qquad$ $V \neq 0$	$\theta \neq 0$ $\qquad$ $\theta \neq 0$



Example 4.6 Determine the slope and deflection of point *A* of the of a cantilever beam *AB* of length *L* and uniform flexural rigidity *EI*. A concentrated force *P* is applied at the free end *of* beam.



(a) A cantilever beam (actual beam)

(b) Conjugate beam (additional beam) corresponding to the actual beam

(c) Free-body diagram for the conjugate beam

(d) Deflections of the cantilever beam (actual beam)

Figure 4.8

Solution: The conjugate beam of the actual beam is shown in Figure 4.8(b). A linearly varying distributed upward *elastic load* with intensity equal to zero at *A* and equal to *PL/EI* at *B.* The free-body diagram for the conjugate beam is shown in Figure 4.8(c). The reactions at *A* of the conjugate beam are given by

$$\overline{V}_A = \frac{1}{2} \times L \times \frac{PL}{EI} = \frac{PL^2}{2EI} \left(\downarrow \right)$$

$$\overline{M}_A = \left(\frac{1}{2} \times L \times \frac{PL}{EI} \right) \times \frac{2L}{3} = \frac{PL^3}{3EI} \left(\curvearrowright \right)$$

The slope at *A* , θ_A and the deflection Δ_A at the free end *A* of the actual beam in Figure 4.8(d) are respectively, equal to the “shearing force” $\overline{V}_A$ and the “bending moment” $\overline{M}_A$ at the fixed end *A* of the conjugate beam in Figure 4.8(d).

$$\theta_A = \frac{PL^2}{2EI} \; (\circlearrowleft)$$

$$\Delta_A = \frac{PL^3}{3EI} \; (\downarrow)$$

Note that Δ_A points downward because $\overline{M}_A$ causes tension in bottom fiber of the beam at *A* (i.e. sagging moment)

Example 4.7 Determine the slope at *A* and deflection of *B* of the beam shown in Figure 4.9(a) using the conjugate beam method.

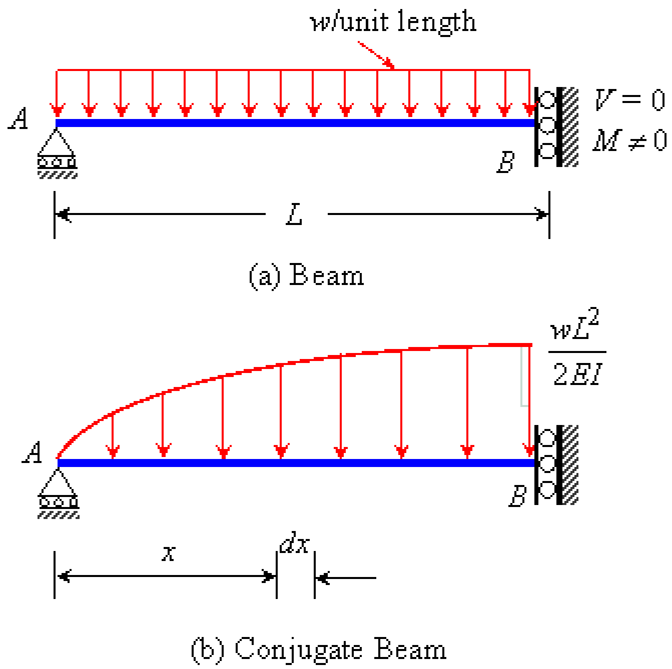


Figure 4.9

Solution: The vertical reaction at *A* in the real beam is given by

$$V_A = wL$$

The bending moment at any point *X* at a distance *x* from *A* is given by

$$M_x = wLx - \frac{1}{2}wx^2$$

The corresponding conjugate beam and loading acting on it are shown in Figure 4.9(b). The loading on the beam varies parabolically with maximum value as $\frac{wL^2}{2EI}$.

The slope at *A* , θ_A in the original beam will be equal to the shear force at *A* in the conjugate beam, thus,

$$\theta_A = -\frac{2}{3} \times L \times \frac{wL^2}{2EI}$$

$$= -\frac{wL^3}{3EI} \quad \text{(clockwise direction)}$$

The deflection of *B* in the real beam will be equal to the bending moment at *B* in conjugate beam i.e.

$$\Delta_B = \frac{2}{3} \times L \times \frac{wL^2}{2EI} \times L - \frac{2}{3} \times L \times \frac{wL^2}{2EI} \times \frac{3L}{8}$$

$$= \frac{5wL^4}{24EI}$$ (downward direction)

Example 4.8 Determine the deflection at the free end of the beam shown in Figure 4.10 using conjugate beam method and verify by moment area method.

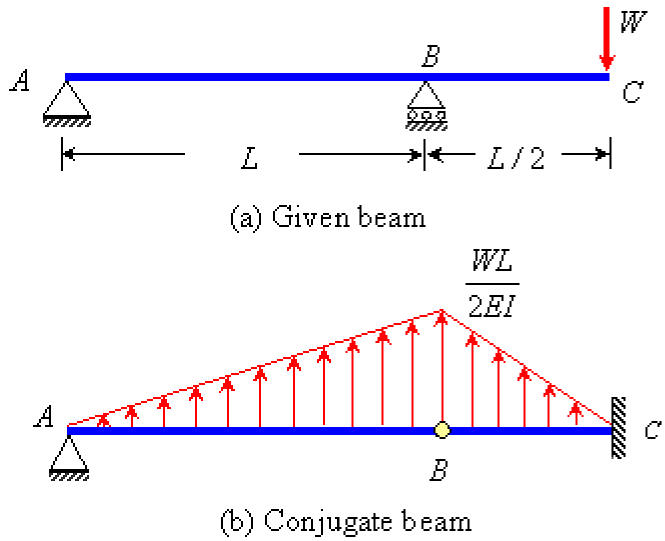


Figure 4.10(a)-(b)

Solution:

(a) Conjugate beam method

The corresponding conjugate beam and loading are shown in Figure 4.10(b). The loading is upward linearly distributed load with maximum value of $\frac{WL}{2EI}$ at *B* . Taking moment about point *B,* the vertical reaction at *A* in the conjugate beam is given by

$$V_A \times L - \frac{1}{2} \times L \times \frac{WL}{2EI} \times \frac{L}{3} = 0$$

$$V_A = \frac{WL^2}{12EI}$$

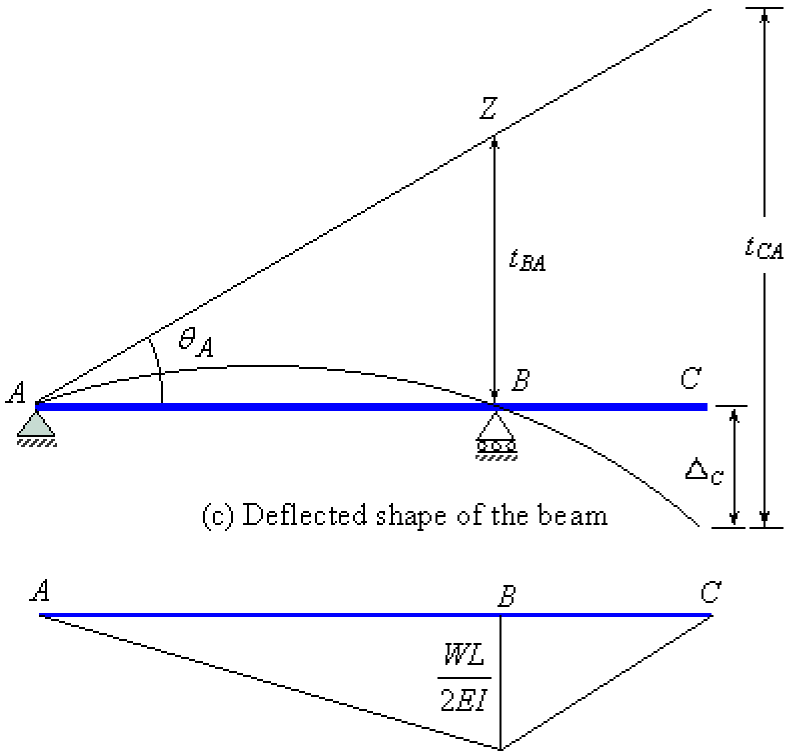
The bending moment at *C* (by taking moment about *C*) is given by

$$\frac{WL^2}{12EI} \times \left(L + \frac{L}{2}\right) - \frac{1}{2} \times \frac{WL}{2EI} \times \left(L + \frac{L}{2}\right)\left(\frac{L + L/2 + L/2}{3}\right) + M_C = 0$$

$$M_C = \frac{WL^3}{8EI}$$ (sagging type)

Hence, the deflection of point *C* will be equal to $\frac{WL^3}{8EI}$ in the downward direction.

(b) Verification by moment-area method



(c) Deflected shape of the beam

(d) Bending moment diagram

Figure 4.10(c)-(d)

Applying second moment area theorem between points *A* and *B* will give the slope at *A* i.e.

$$\theta_A \times L = t_{BA}$$

$$= \frac{1}{2} \times L \times \frac{WL}{2EI} \times \frac{L}{3}$$

$$\theta_A = \frac{WL^2}{12EI}$$

Further, applying moment area theorem between point *A* and *C*

$$t_{CA} = \frac{1}{2} \times \left(L + \frac{L}{2}\right) \times \frac{WL}{2EI} \times \left(\frac{L + L/2 + L/2}{3}\right)$$

$$t_{CA} = \frac{WL^3}{4EI}$$

$$= -\frac{wL^3}{3EI}$$ (clockwise direction)

Since

$$t_{CA} = \theta_A \times \left(L + \frac{L}{2}\right) + \Delta_C$$

$$\frac{WL^3}{4EI} = \frac{WL^2}{12EI} \times \left(L + \frac{L}{2}\right) + \Delta_C$$

$$\Delta_C = \frac{WL^3}{8EI}$$ (downward direction)

Recap

In this course you have learnt the following

- Computation of deflection using conjugate beam method.