

Module 4 : Deflection of Structures

Lecture 1 : Moment Area Method

Objectives

In this course you will learn the following

- Importance of computation of deflection.
- Computation of deflection using moment area method.

4.1 Introduction

When a structure is subjected to the action of applied loads each member undergoes deformation due to which the axis of structure is deflected from its original position. The deflections also occur due to temperature variations and lack-of-fit of members. The deflections of structures are important for ensuring that the designed structure is not excessively flexible. The large deformations in the structures can cause damage or cracking of non-structural elements. The deflection in beams is dependent on the acting bending moments and its flexural stiffness. The computation of deflections in structures is also required for solving the statically indeterminate structures.

In this chapter, several methods for computing deflection of structures are considered.

4.2 Moment Area Method

The moment-area method is one of the most effective methods for obtaining the bending displacement in beams and frames. In this method, the area of the bending moment diagrams is utilized for computing the slope and or deflections at particular points along the axis of the beam or frame. Two theorems known as the moment area theorems are utilized for calculation of the deflection. One theorem is used to calculate the change in the slope between two points on the elastic curve. The other theorem is used to compute the vertical distance (called tangential deviation) between a point on the elastic curve and a line tangent to the elastic curve at a second point.

Consider Figure 4.1 showing the elastic curve of a loaded simple beam. On the elastic curve tangents are drawn on points A and B . Total angle between the two tangents is denoted as $\Delta\theta_{AB}$. In order to find out $\Delta\theta_{AB}$, consider the incremental change in angle $d\theta$ over an infinitesimal segment dx located at a distance of x from point B . The radius of curvature and bending moment for any section of the beam is given by the usual bending equation.

$$\frac{M}{I} = \frac{E}{R} \quad (4.1)$$

where R is the radius of curvature; E is the modulus of elasticity; I is the moment of inertia; and M denotes the bending moment.

The elementary length dx and the change in angle $d\theta$ are related as,

$$dx = d\theta \times R \quad (4.2)$$

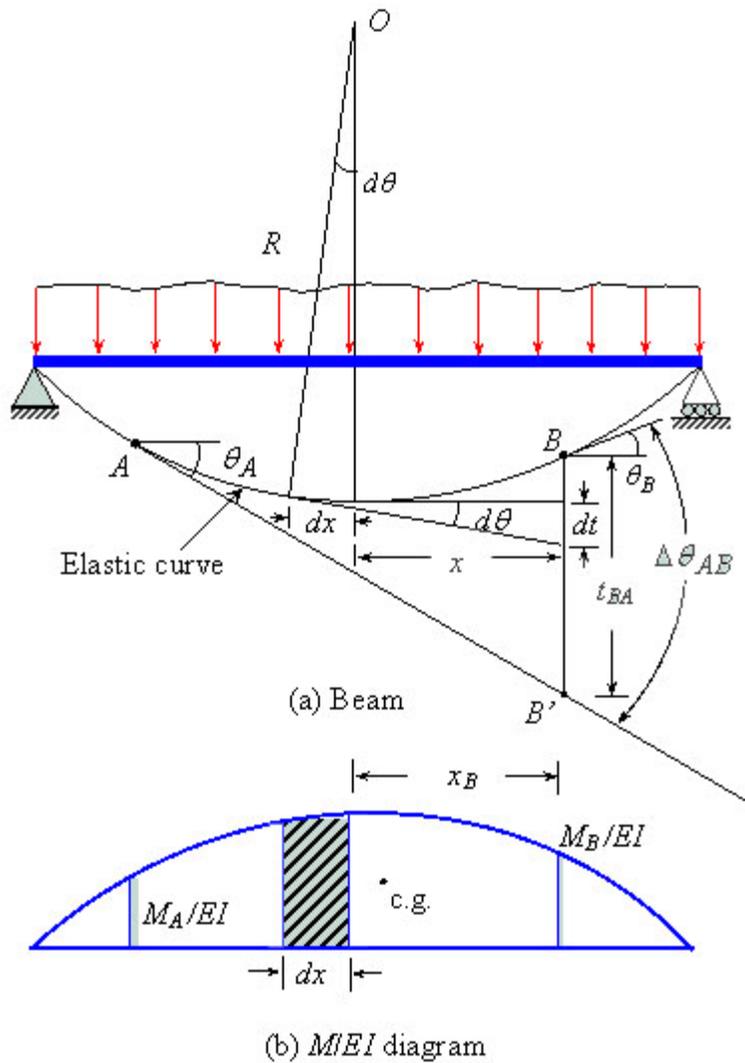


Figure 4.1

Substituting R from Eq. (4.2) in Eq. (4.1)

$$d\theta = \frac{M}{EI} dx \quad (4.3)$$

The total angle change $\Delta\theta_{AB}$ can be obtained by integrating Eq. (4.3) between points A and B which is expressed as

$$\Delta\theta_{AB} = \int_A^B d\theta = \int_A^B \frac{M}{EI} dx \quad (4.4a)$$

or,

$$\theta_B - \theta_A = \text{Area of } M / EI \text{ diagram between } A \text{ and } B \quad (4.4b)$$

The difference of slope between any two points on a continuous elastic curve of a beam is equal to the area under the M / EI curve between these points.

The distance dt along the vertical line through point B is nearly equal to.

$$dt = x \times d\theta \quad (4.5)$$

Integration of dt between points A and B yield the vertical distance t_{BA} between the point B and the tangent from point A on the elastic curve. Thus,

$$t_{BA} = \int_A^B x d\theta = \int_A^B \frac{Mx}{EI} dx \quad (4.6)$$

since the quantity Mdx/EI represents an infinitesimal area under the M/EI diagram and distance x from that area to point B , the integral on right hand side of Eq. (4.6) can be interpreted as moment of the area under the M/EI diagram between points A and B about point B . This is the second moment area theorem.

If A and B are two points on the deflected shape of a beam, the vertical distance of point B from the tangent drawn to the elastic curve at point A is equal to the moment of bending moment diagram area between the points A and B about the vertical line from point B , divided by EI .

Sign convention used here can be remembered keeping the simply supported beam of Figure 4.1 in mind. A sagging moment is the positive bending moment diagram and has positive area. Slopes are positive if measured in the anti-clockwise direction. Positive deviation t_{BA} indicates that the point B lies above the tangent from the point A .

Example 4.1 Determine the end slope and deflection of the mid-point C in the beam shown below using moment area method.

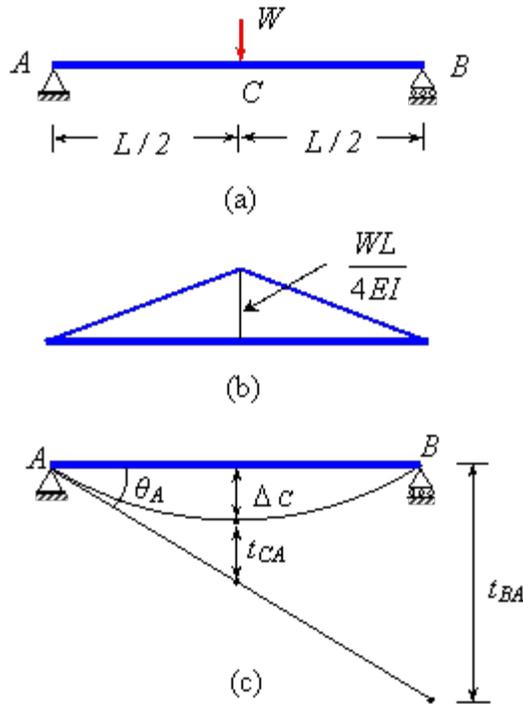


Figure 4.2

Solution: The M/EI diagram of the beam is shown in Figure 4.2(a). The slope at A , θ_A can be obtained by computing the t_{BA} using the second moment area theorem i.e.

$$\theta_A \times L = t_{BA}$$

$$\theta_A = \frac{1}{L} \times \left(\frac{1}{2} \times \frac{WL}{4EI} \times L \times \frac{L}{2} \right) = \frac{WL^2}{16EI} \text{ (clockwise direction)}$$

The slope at B can be obtained by using the first moment area theorem between points A and B i.e.

$$\theta_B - \theta_A = \Delta\theta_{AB}$$

$$\theta_B - \theta_A = \frac{1}{2} \times \frac{WL}{4EI} \times L = \frac{WL^2}{8EI}$$

$$\theta_B = \frac{WL^2}{8EI} - \frac{WL^2}{16EI} = \frac{WL^2}{16EI} \text{ (anti-clockwise)}$$

(It is to be noted that the $\theta_A = -\frac{WL^2}{16EI}$. The negative sign is because of the slope being in the clockwise direction. As per sign convention a positive slope is in the anti-clockwise direction)

The deflection at the centre of the beam can be obtained with the help of the second moment area theorem between points A and C i.e.

$$\theta_A \times \frac{L}{2} = \Delta_C + t_{CA}$$

$$\frac{WL^2}{16EI} \times \frac{L}{2} = \Delta_C + \left(\frac{1}{2} \times \frac{WL}{4EI} \times \frac{L}{2} \times \frac{L}{6} \right)$$

$$\Delta_C = \frac{WL^3}{48EI} \text{ (downward direction)}$$

Example 4.2 Using the moment area method, determine the slope at B and C and deflection at C of the cantilever beam as shown in Figure 4.3(a). The beam is subjected to uniformly distributed load over entire length and point load at the free end.

Solution: The moment curves produced by the concentrated load, W and the uniformly distributed load, w are plotted separately and divided by EI (refer Figures 4.3(b) and (c)). This results in the simple geometric shapes in which the area and locations of their centroids are known.

Since the end A is fixed, therefore, $\theta_A = 0$. Applying the first moment-area theorem between points A and C

$$\theta_C - \theta_A = \Delta\theta_{AC}$$

$$\theta_C - \theta_A = - \left(\frac{1}{2} \times L \times \frac{WL}{EI} + \frac{1}{3} \times L \times \frac{wL^2}{2EI} \right) \text{ (negative sign is due to hogging moment)}$$

$$\theta_C = - \left(\frac{WL^2}{2EI} + \frac{wL^3}{6EI} \right) \text{ (clockwise direction)}$$

The slope at B can be obtained by applying the first moment area theorem between points B and C i.e.

$$\theta_C - \theta_B = \Delta\theta_{BC}$$

$$\theta_B = \theta_C - \Delta\theta_{BC}$$

$$\theta_B = - \left(\frac{WL^2}{2EI} + \frac{wL^3}{6EI} \right) - \left(- \frac{1}{2} \times \frac{L}{2} \times \frac{WL}{2EI} - \frac{1}{3} \times \frac{L}{2} \times \frac{wL^2}{8EI} \right)$$

$$\theta_B = - \left(\frac{3WL^2}{8EI} + \frac{7wL^3}{48EI} \right) \text{ (clockwise direction)}$$

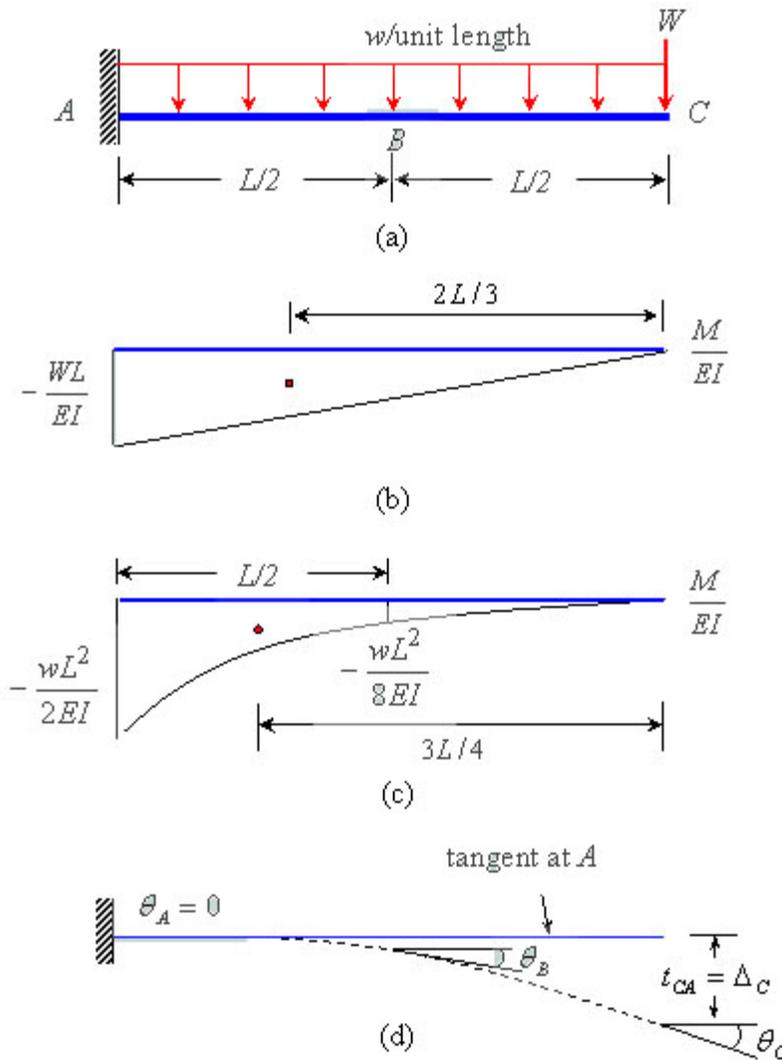


Figure 4.3

The deflection at C is equal to the tangential deviation of point C from the tangent to the elastic curve at A (see Figure 4.3(d)).

$\Delta_C = t_{CA} =$ moment of areas under M/EI curves between A and C in Figures 4.3(b) and (c) about C

$$\begin{aligned}
 &= \frac{1}{2} \times L \times \frac{WL}{EI} \times \frac{2L}{3} + \frac{1}{3} \times L \times \frac{wL^2}{2EI} \times \frac{3L}{4} \\
 &= \frac{WL^3}{3EI} + \frac{wL^4}{8EI} \quad (\text{downward direction})
 \end{aligned}$$

Example 4.3 Determine the end-slopes and deflection at the center of a non-prismatic simply supported beam. The beam is subjected to a concentrated load at the center.

Solution: The M/EI diagram of the beam is shown in Figure 4.4(b).

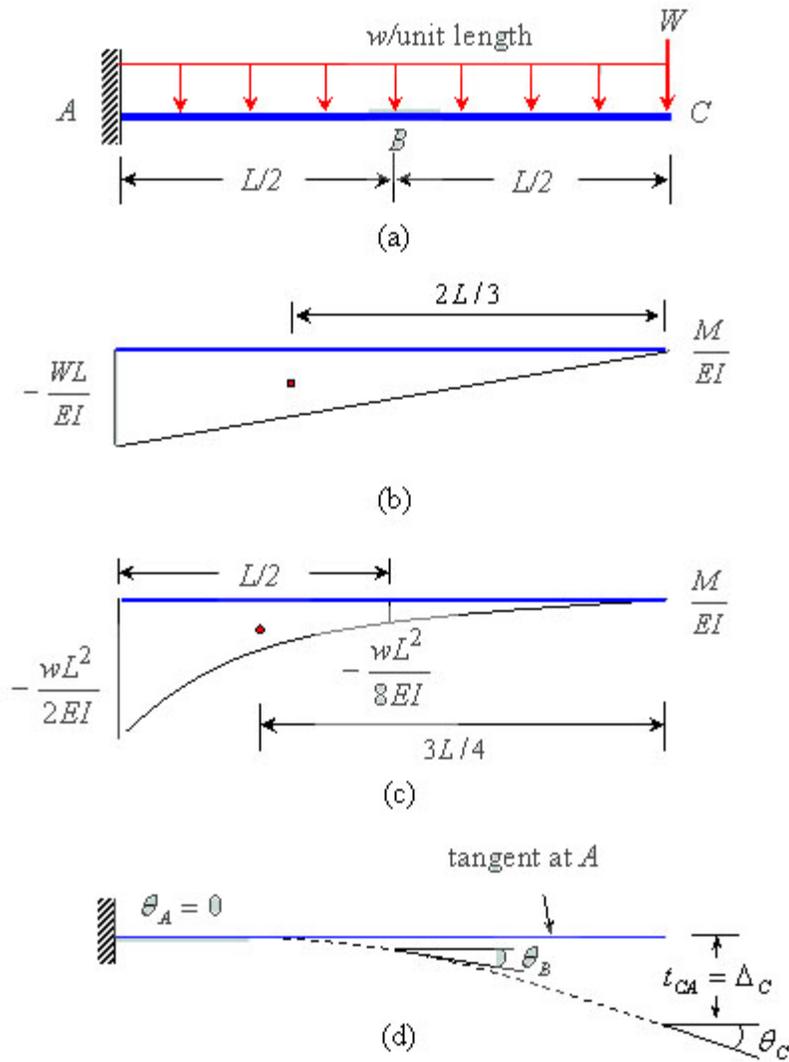


Figure 4.3

Applying second moment-area theorem between points A and B ,

$$t_{BA} = \frac{1}{2} \times \frac{L}{2} \times \frac{PL}{4EI} \left(\frac{L}{2} + \frac{L}{6} \right) + \frac{1}{2} \times \frac{PL}{8EI} \times \frac{L}{2} \times \left(\frac{2}{3} \times \frac{L}{2} \right)$$

$$-\theta_{AL} = \frac{PL^3}{24EI} + \frac{PL^3}{96EI} = \frac{5PL^3}{96EI}$$

$$\theta_A = -\frac{5PL^2}{96EI} \text{ (clockwise direction)}$$

Applying first moment area theorem between A and C .

$$\theta_C - \theta_A = \frac{1}{2} \times \frac{L}{2} \times \frac{PL}{4EI}$$

$$\theta_C = \frac{PL^2}{16EI} - \frac{5PL^2}{96EI} = \frac{PL^2}{96EI} \text{ (anti-clockwise direction)}$$

Applying second moment area theorem between A and C .

$$t_{CA} = \frac{PL^2}{16EI} \times \frac{L}{6} = \frac{PL^3}{96EI}$$

$$\left(\theta_A \times \frac{L}{2}\right) - \Delta_C = \frac{PL^3}{96EI}$$

$$\Delta_C = \frac{5PL^2}{96EI} \times \frac{L}{2} - \frac{PL^3}{96EI} = \frac{PL^3}{64EI} \text{ (downward direction)}$$

Example 4.4 Determine the slope and deflection at the hinge of the beam shown in the Figure 4.5 (a).

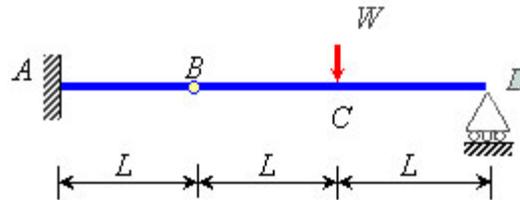


Figure 4.5(a)

Solution: The bending moment diagram is shown in Figure 4.5(b).

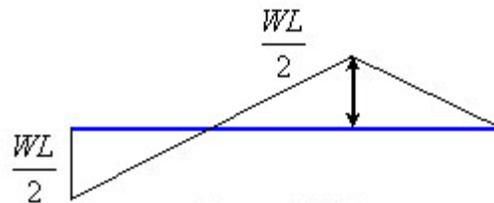


Figure 4.5(b)

Since the end A is fixed, therefore, $\theta_A = 0$. Applying the first moment-area theorem between points A and B (refer Figure 4.5(c))

$$\theta_{BA} - \theta_A = -\frac{1}{2} \times \frac{WL}{2} \times L \times \frac{1}{EI}$$

$$\theta_{BA} = -\frac{WL^2}{4EI} \text{ (clockwise direction)}$$

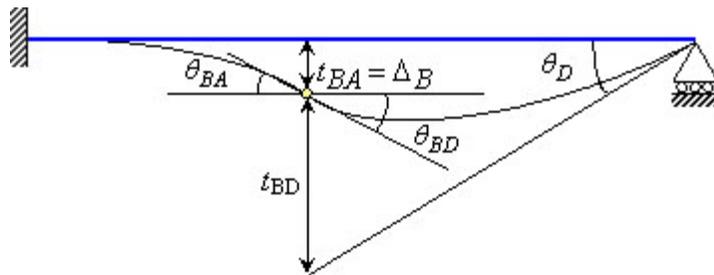


Figure 4.5(c)

Applying second moment area theorem between points A and B ,

$$\Delta_B = t_{BA} = \frac{1}{2} \times \frac{WL}{2} \times L \times \frac{1}{EI} \times \frac{2L}{3}$$

$$= \frac{WL^3}{6EI} \text{ (downward direction)}$$

Applying second moment area theorem between points B and D ,

$$\theta_D \times 2L = (t_{BA} + t_{BD})$$

$$\theta_D = \frac{1}{2L} \times \left(\frac{WL^3}{6EI} + \frac{1}{2} \times \frac{WL}{2} \times 2L \times \frac{1}{EI} \times L \right)$$

$$\theta_D = \frac{WL^2}{3EI} \text{ (anti-clockwise direction)}$$

From the first moment area theorem between points B and D

$$\theta_D - \theta_{BD} = \frac{1}{2} \times \frac{WL}{2} \times 2L \times \frac{1}{EI}$$

$$\theta_{BD} = \frac{WL^2}{3EI} - \frac{WL^2}{2EI}$$

$$\theta_{BD} = -\frac{WL^2}{6EI} \text{ (clockwise direction)}$$

Example 4.5 Determine the vertical deflection and slope of point C of the rigid-jointed plane frame shown in the Figure 4.6(a).

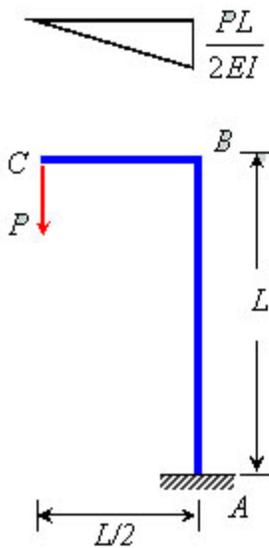


Figure 4.6(a)

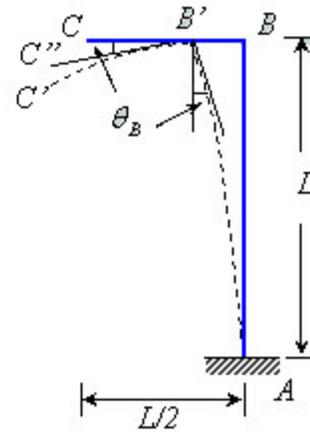
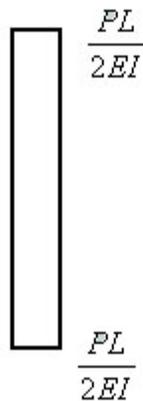


Figure 4.6(b)

Solution: The M/EI and deflected shape of the frame are shown in the Figures 4.6(a) and (b), respectively. As the point A is fixed implying that $\theta_A = 0$. Applying first moment area theorem between points A and B ,

$$\theta_A - \theta_B = -\frac{PL}{2EI} \times L \text{ (looking from the left side)}$$

$$\theta_B = \frac{PL^2}{2EI} \text{ (anti-clockwise direction)}$$

Applying second moment area theorem between points B and C

$$t_{CB} = -\frac{1}{2} \times \frac{PL}{2EI} \times \frac{L}{2} \times \left(\frac{2}{3} \times \frac{L}{2} \right) = -\frac{PL^3}{24EI}$$

The vertical displacement of point C

$$\Delta_C = -\theta_B \times \frac{L}{2} + t_{CB}$$

$$\Delta_C = -\theta_B \times \frac{L}{2} + t_{CB} \text{ (downward direction)}$$

Applying first moment area theorem between point B and C

$$\theta_B - \theta_C = -\frac{1}{2} \times \frac{PL}{2EI} \times L/2$$

$$\theta_C = \frac{5PL^2}{8EI} \text{ (anti-clockwise direction)}$$

Recap

In this course you have learnt the following

- Importance of computation of deflection.
- Computation of deflection using moment area method.