Chapter 2

Fundamental Relations of Traffic Flow

2.1 Overview

Speed is one of the basic parameters of traffic flow and time mean speed and space mean speed are the two representations of speed. Time mean speed and space mean speed and the relationship between them will be discussed in detail in this chapter. The relationship between the fundamental parameters of traffic flow will also be derived. In addition, this relationship can be represented in graphical form resulting in the fundamental diagrams of traffic flow.

2.2 Time mean speed (v_t)

As noted earlier, time mean speed is the average of all vehicles passing a point over a duration of time. It is the simple average of spot speed. Time mean speed v_t is given by,

$$v_t = \frac{1}{n}\sum_{i=1}^{n} v_i, \tag{2.1}$$

where v_i is the spot speed of i^{th} vehicle, and n is the number of observations. In many speed studies, speeds are represented in the form of frequency table. Then the time mean speed is given by,

$$v_t = \frac{\sum_{i=1}^{n} q_i v_i}{\sum_{i=1}^{n} q_i}, \tag{2.2}$$

where q_i is the number of vehicles having speed v_i, and n is the number of such speed categories.

2.3 Space mean speed (v_s)

The space mean speed also averages the spot speed, but spatial weightage is given instead of temporal. This is derived as below. Consider unit length of a road, and let v_i is the spot speed of i^{th} vehicle. Let t_i is the time the vehicle takes to complete unit distance and is given by $\frac{1}{v_i}$. If there are n such vehicles, then the average travel time t_s is given by,

$$t_s = \frac{\Sigma t_i}{n} = \frac{1}{n}\Sigma\frac{1}{v_i}. \tag{2.3}$$

If t_{av} is the average travel time, then average speed $v_s = \frac{1}{t_s}$. Therefore, from the above equation,

$$v_s = \frac{n}{\sum_{i=1}^{n}\frac{1}{v_i}}. \tag{2.4}$$

This is simply the harmonic mean of the spot speed. If the spot speeds are expressed as a frequency table, then,

$$v_s = \frac{\sum_{i=1}^{n} q_i}{\sum_{i=1}^{n}\frac{q_i}{v_i}} \tag{2.5}$$

where q_i vehicle will have v_i speed and n_i is the number of such observations.

Numerical Example

If the spot speeds are 50, 40, 60, 54 and 45, then find the time mean speed and space mean speed.

Solution Time mean speed v_t is the average of spot speed. Therefore, $v_t = \frac{\Sigma v_i}{n} = \frac{50+40+60+54+45}{5} = \frac{249}{5} = 49.8$. Space mean speed is the harmonic mean of spot speed. Therefore, $v_s = \frac{n}{\Sigma\frac{1}{v_i}} = \frac{5}{\frac{1}{50}+\frac{1}{40}+\frac{1}{60}+\frac{1}{54}+\frac{1}{45}} = \frac{5}{0.12} = 48.82$.

Numerical Example

The results of a speed study is given in the form of a frequency distribution table. Find the time mean speed and space mean speed.

speed range	frequency
2-5	1
6-9	4
10-13	0
14-17	7

Solution The time mean speed and space mean speed can be found out from the frequency table given below. First, the average speed is computed, which is the mean of the speed range. For example, for the first speed range, average speed, $v_i = \frac{2+5}{2} = 3.5$ seconds. The volume of flow q_i for that speed range is same as the frequency. The terms $v_i.q_i$ and $\frac{q_i}{v_i}$ are also tabulated,

No.	speed range	average speed (v_i)	volume of flow (q_i)	$v_i q_i$	$\frac{q_i}{v_i}$
1	2-5	3.5	1	3.5	2.29
2	6-9	7.5	4	30.0	0.54
3	10-13	11.5	0	0	0
4	14-17	15.5	7	108.5	0.45
	total		12	142	3.28

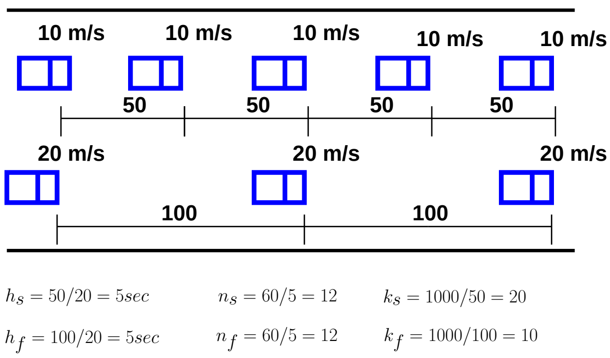

Figure 2:1: Illustration of relation between time mean speed and space mean speed

and their summations given in the last row. Time mean speed can be computed as, $v_t = \frac{\Sigma q_i v_i}{\Sigma q_i} = \frac{142}{12} = 11.83$. Similarly, space mean speed can be computed as, $v_s = \frac{\Sigma q_i}{\Sigma \frac{q_i}{v_i}} = \frac{12}{3.28} = 3.65$.

2.4 Illustration of mean speeds

In order to understand the concept of time mean speed and space mean speed, following illustration will help. Let there be a road stretch having two sets of vehicle as in figure 2:1. The first vehicle is traveling at 10m/s with 50 m spacing, and the second set at 20m/s with 100 m spacing. Therefore, the headway of the slow vehicle h_s will be 50 m divided by 10 m/s which is 5 sec. Therefore, the number of slow moving vehicles observed at A in one hour n_s will be $60/5 = 12$ vehicles. The density K is the number of vehicles in 1 km, and is the inverse of spacing. Therefore, $K_s = 1000/50 = 20$ vehicles/km. Therefore, by definition, time mean speed v_t is given by $v_t = \frac{12\times10+12\times20}{24} = 15\ m/s$. Similarly, by definition, space mean speed is the mean of vehicle speeds over time. Therefore, $v_s = \frac{20\times10+10\times20}{30} = 13.3\ m/s$. This is same as the harmonic mean of spot speeds obtained at location A; ie $v_s = \frac{24}{12\times\frac{1}{10}+12\times\frac{1}{20}} = 13.3\ m/s$. It may be noted that since harmonic mean is always lower than the arithmetic mean, and also as observed, space mean speed is always lower than the time mean speed. In other words, space mean speed weights slower vehicles more heavily as they occupy the road stretch for longer duration of time. For this reason, in many fundamental traffic equations, space mean speed is preferred over time mean speed.

2.5 Relation between time mean speed and space mean speed

The relation between time mean speed(v_t) and space mean speed(v_s) is given by the following relation:

$$v_t = v_s + \frac{\sigma^2}{v_s} \tag{2.6}$$

where,σ^2 is the standard deviation of the spot speed. The derivation of the formula is given in the next subsection. The standard deviation(σ^2) can be computed in the following equation:

$$\sigma^2 = \frac{\Sigma q_i v_i^2}{\Sigma q_i} - (v_t)^2 \tag{2.7}$$

where,q_i is the frequency of the vehicle having v_i speed.

2.5.1 Derivation of the relation

The relation between time mean speed and space mean speed can be derived as below. Consider a stream of vehicles with a set of sub-stream flow q_1, q_2, $\ldots q_i$, $\ldots q_n$ having speed v_1,v_2, $\ldots v_i$, $\ldots v_n$. The fundamental relation between flow(q), density(k) and mean speed v_s is,

$$q = k \times v_s \tag{2.8}$$

Therefore for any sub-stream q_i, the following relationship will be valid.

$$q_i = k_i \times v_i \tag{2.9}$$

The summation of all sub-stream flows will give the total flow q:

$$\Sigma q_i = q. \tag{2.10}$$

Similarly the summation of all sub-stream density will give the total density k.

$$\Sigma k_i = k. \tag{2.11}$$

Let f_i denote the proportion of sub-stream density k_i to the total density k,

$$f_i = \frac{k_i}{k}. \tag{2.12}$$

Space mean speed averages the speed over space. Therefore, if k_i vehicles has v_i speed, then space mean speed is given by,

$$v_s = \frac{\Sigma k_i v_i}{k}. \tag{2.13}$$

Time mean speed averages the speed over time. Therefore,

$$v_t = \frac{\Sigma q_i v_i}{q}. \tag{2.14}$$

Substituting 2.9, v_t can be written as,

$$v_t = \frac{\Sigma k_i {v_i}^2}{q} \tag{2.15}$$

Rewriting the above equation and substituting 2.12, and then substituting 2.8, we get,

$$\begin{aligned} v_t &= k\Sigma \frac{k_i}{k} v_i^2 \\ &= \frac{k\Sigma f_i {v_i}^2}{q} \\ &= \frac{\Sigma f_i {v_i}^2}{v_s} \end{aligned}$$

By adding and subtracting v_s and doing algebraic manipulations, v_t can be written as,

$$v_t = \frac{\Sigma f_i (v_s + (v_i - v_s))^2}{v_s} \tag{2.16}$$

$$= \frac{\Sigma f_i (v_s)^2 + (v_i - v_s)^2 + 2.v_s.(v_i - v_s)}{v_s} \tag{2.17}$$

$$= \frac{\Sigma f_i {v_s}^2}{v_s} + \frac{\Sigma f_i (v_i - v_s)^2}{v_s} + \frac{2.v_s.\Sigma f_i (v_i - v_s)}{v_s} \tag{2.18}$$

The third term of the equation will be zero because $\Sigma f_i(v_i - v_s)$ will be zero, since v_s is the mean speed of v_i. The numerator of the second term gives the standard deviation of v_i. Σf_i by definition is 1.Therefore,

$$v_t = v_s \Sigma f_i + \frac{\sigma^2}{v_s} + 0 \tag{2.19}$$

$$= v_s + \frac{\sigma^2}{v_s} \tag{2.20}$$

Hence, time mean speed is space mean speed plus standard deviation of the spot speed divided by the space mean speed. Time mean speed will be always greater than space mean speed since standard deviation cannot be negative. If all the speed of the vehicles are the same, then spot speed, time mean speed and space mean speed will also be same.

No.	speed range $v^l < v < v^u$	mid interval $v_i = \frac{v_l+v_u}{2}$	flow q_i	$q_i v_i$	v_i^2	$q_i v_i^2$	q_i/v_i
1	0-10	5	6	30	25	150	6/5
2	10-20	15	16	240	225	3600	16/15
3	20-30	20	24	600	625	15000	24/25
4	30-40	25	25	875	1225	30625	25/35
5	40-50	30	17	765	2025	34425	17/45
	total		88	2510		83800	4.3187

Numerical Example

For the data given below,compute the time mean speed and space mean speed. Also verify the relationship between them. Finally compute the density of the stream.

speed range	frequency
0-10	5
10-20	15
20-30	20
30-40	25
40-50	30

Solution The solution of this problem consist of computing the time mean speed $v_t = \frac{\Sigma q_i v_i}{\Sigma q_i}$,space mean speed $v_s = \frac{\Sigma q_i}{\Sigma \frac{q_i}{v_i}}$,verifying their relation by the equation $v_t = v_s + \frac{\sigma^2}{v_s}$,and using this to compute the density. To verify their relation, the standard deviation also need to be computed $\sigma^2 = \frac{\Sigma q v^2}{\Sigma q} - v_t^2$. For convenience,the calculation can be done in a tabular form as shown in table 2.5.1.

The time mean speed(v_t) is computed as:

$$\begin{aligned} v_t &= \frac{\Sigma q_i v_i}{\Sigma q_i} \\ &= \frac{2510}{88} = 28.52 \end{aligned}$$

The space mean speed can be computed as:

$$\begin{aligned} v_s &= \frac{\Sigma q_i}{\frac{\Sigma q_i}{v_i}} \\ &= \frac{88}{4.3187} = 20.38 \end{aligned}$$

The standard deviation can be computed as:

$$\begin{aligned} \sigma^2 &= \frac{\Sigma q v^2}{\Sigma q} - v_t^2 \\ &= \frac{83800}{88} - 28.52^2 = 138.727 \end{aligned}$$

The time mean speed can also v_t can also be computed as:

$$v_t = v_s + \frac{\sigma^2}{v_s} = 20.38 + \frac{138.727}{20.38} = 27.184$$

The density can be found as:

$$k = \frac{q}{v} = \frac{88}{20.38} = 4.3 \text{ vehicle/km}$$

2.6 Fundamental relations of traffic flow

The relationship between the fundamental variables of traffic flow, namely speed, volume, and density is called the fundamental relations of traffic flow. This can be derived by a simple concept. Let there be a road with length v km, and assume all the vehicles are moving with v km/hr.(Fig 2:2). Let the number of vehicles counted by an observer at A for one hour be n_1. By definition, the number of vehicles counted in one hour is flow(q). Therefore,

$$n_1 = q. \tag{2.21}$$

Similarly, by definition, density is the number of vehicles in unit distance. Therefore number of vehicles n_2 in a road stretch of distance v_1 will be density $\times$ distance.Therefore,

$$n_2 = k \times v. \tag{2.22}$$

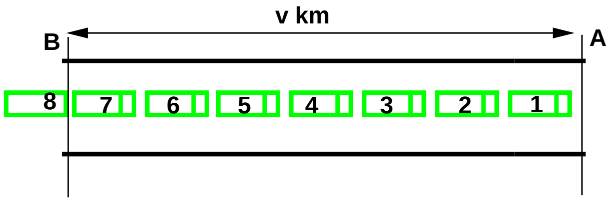

Figure 2:2: Illustration of relation between fundamental parameters of traffic flow

Since all the vehicles have speed v, the number of vehicles counted in 1 hour and the number of vehicles in the stretch of distance v will also be same.(ie $n_1 = n_2$). Therefore,

$$q = k \times v. \tag{2.23}$$

This is the fundamental equation of traffic flow. Please note that, v in the above equation refers to the space mean speed will also be same.

2.7 Fundamental diagrams of traffic flow

The relation between flow and density, density and speed, speed and flow, can be represented with the help of some curves. They are referred to as the fundamental diagrams of traffic flow. They will be explained in detail one by one below.

2.7.1 Flow-density curve

The flow and density varies with time and location. The relation between the density and the corresponding flow on a given stretch of road is referred to as one of the fundamental diagram of traffic flow. Some characteristics of an ideal flow-density relationship is listed below:

1. When the density is zero, flow will also be zero,since there is no vehicles on the road.

2. When the number of vehicles gradually increases the density as well as flow increases.

3. When more and more vehicles are added, it reaches a situation where vehicles can't move. This is referred to as the jam density or the maximum density. At jam density, flow will be zero because the vehicles are not moving.

4. There will be some density between zero density and jam density, when the flow is maximum. The relationship is normally represented by a parabolic curve as shown in figure 2:3

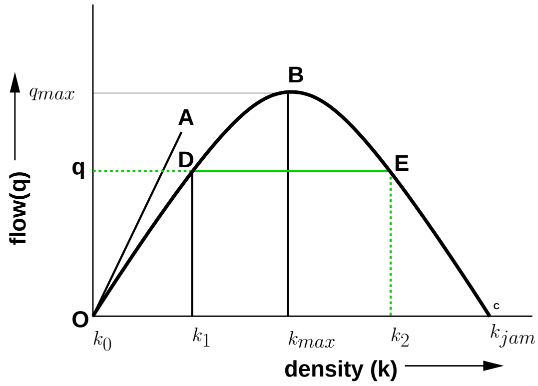

Figure 2:3: Flow density curve

The point O refers to the case with zero density and zero flow. The point B refers to the maximum flow and the corresponding density is k_{max}. The point C refers to the maximum density k_{jam} and the corresponding flow is zero. OA is the tangent drawn to the parabola at O, and the slope of the line OA gives the mean free flow speed, ie the speed with which a vehicle can travel when there is no flow. It can also be noted that points D and E correspond to same flow but has two different densities. Further, the slope of the line OD gives the mean speed at density k_1 and slope of the line OE will give mean speed at density k_2. Clearly the speed at density k_1 will be higher since there are less number of vehicles on the road.

2.7.2 Speed-density diagram

Similar to the flow-density relationship, speed will be maximum, referred to as the free flow speed, and when the density is maximum, the speed will be zero. The most simple assumption is that this variation of speed with density is linear as shown by the solid line in figure 2:4. Corresponding to the zero density, vehicles will be flowing with their desire speed, or free flow speed. When the density is jam density, the speed of the vehicles becomes zero. It is also possible to have non-linear relationships as shown by the dotted lines. These will be discussed later.

2.7.3 Speed flow relation

The relationship between the speed and flow can be postulated as follows. The flow is zero either because there is no vehicles or there are too many vehicles so that they cannot move. At maximum flow, the speed will be in between zero and free flow speed. This relationship is shown in figure 2:5. The maximum flow q_{max} occurs at speed u. It is possible to have two

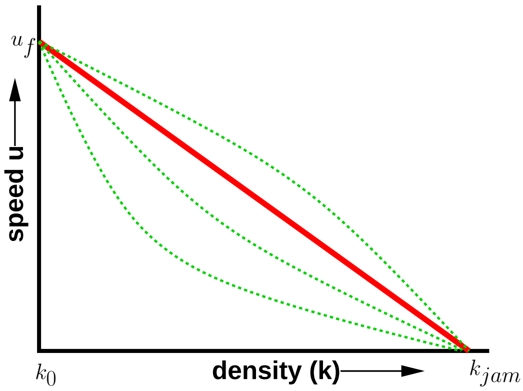

Figure 2:4: Speed-density diagram

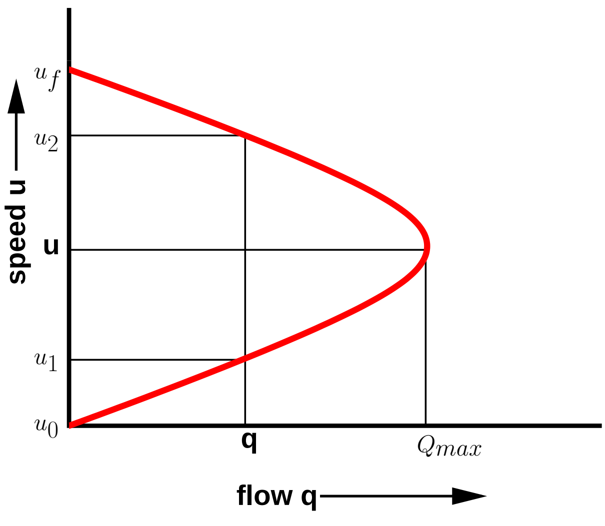

Figure 2:5: Speed-flow diagram

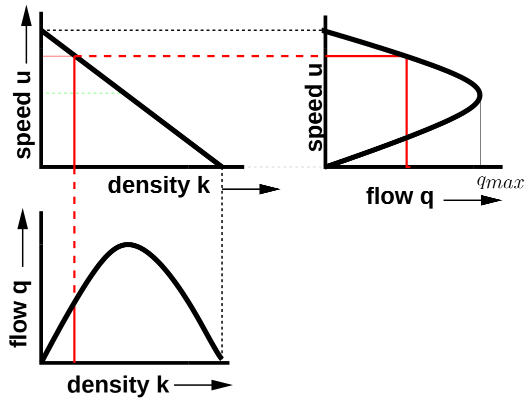

Figure 2:6: Fundamental diagram of traffic flow

different speeds for a given flow.

2.7.4 Combined diagrams

The diagrams shown in the relationship between speed-flow, speed-density, and flow-density are called the fundamental diagrams of traffic flow. These are as shown in figure 2:6. One could observe the inter-relationship of these diagrams.

2.8 Summary

Time mean speed and space mean speed are two important measures of speed. It is possible to have a relation between them and was derived in this chapter. Also, time mean speed will be always greater than or equal to space mean speed. The fundamental diagrams of traffic flow are vital tools which enables analysis of fundamental relationships. There are three diagrams - speed-density, speed-flow and flow-density. They can be together combined in a single diagram as discussed in the last section of the chapter.

2.9 References

1. L. R Kadiyali. *Traffic Engineering and Transportation Planning*. Khanna Publishers, New Delhi, 1987.