Solutions

7 Vector and tensor analysis:

1.

2.

3. (a) The easiest way to solve this problem is the following. Consider the vector:

$$A_i = \epsilon_{ijk}\partial_j\partial_k\phi \tag{16}$$

Since $\epsilon_{ijk} = -\epsilon_{ikj}$,

$$-A_i = \epsilon_{ikj}\partial_j\partial_k\phi = \epsilon_{ikj}\partial_k\partial_j\phi \tag{17}$$

since the derivatives can be interchanged. But the second term on the right side is also equal to A_i, since the j and k are summed over. Therefore, we get an equation of the form $A_i = -A_i$, implying that $A_i = 0$.

The physical interpretation is that $\nabla\phi$ is in the direction $\perp$ to lines of constant ϕ. However, $\nabla\times\nabla\phi$ involves derivatives in the plane $\perp$ to $\nabla\phi$, which is tangent to surfaces of constant ϕ, so these derivatives are zero.

(b) First prove that $\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$. Since $\epsilon_{ijk}\epsilon_{klm}$ is a real vector, and it is isotropic, it has to be of the form:

$$\epsilon_{ijk}\epsilon_{klm} = A\delta_{il}\delta_{jm} + B\delta_{im}\delta_{jl} + C\delta_{ij}\delta_{lm} \tag{18}$$

Multiplying by $\delta_{il}\delta_{jm}$, $\delta_{im}\delta_{jl}$ and $\delta_{ij}\delta_{ml}$ respectively, we get the following equations:

$$\begin{aligned} \epsilon_{ijk}\epsilon_{kij} &= A\delta_{ii}\delta_{jj} + B\delta_{ij}\delta_{ij} + C\delta_{ij}\delta_{ij} \\ &\rightarrow 6 = 9A + 3B + 3C \\ \epsilon_{ijk}\epsilon_{kji} &= A\delta_{ij}\delta_{ij} + B\delta_{ii}\delta_{jj} + C\delta_{ij}\delta_{ij} \\ &\rightarrow -6 = 3A + 9B + 3C \\ \epsilon_{iik}\epsilon_{kll} &= A\delta_{jl}\delta_{jl} + B\delta_{jm}\delta_{jm} + C\delta_{ii}\delta_{ll} \\ &\rightarrow 0 = 3A + 3B + 9C \end{aligned} \tag{19}$$

The solutions to these equations are $A = 1$, $B = -1$ and $C = 0$, and hence we obtain the above identity.

This can be easily used to prove:

$$\begin{aligned} \nabla \times \nabla \times \mathbf{u} &= \epsilon_{ijk}\epsilon_{klm}\partial_j\partial_l u_m \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_j\partial_l u_m \\ &= \partial_i\partial_j u_j - \partial_j^2 u_i \\ &= \nabla(\nabla\cdot\mathbf{u}) - \nabla^2\mathbf{u} \end{aligned} \tag{20}$$

(c) If $B = S_{ij}A_{ij}$, (B is a scalar), then we find that $B = -S_{ji}A_{ji}$ (since $S_{ij} = S_{ji}$ and $A_{ij} = -A_{ji}$). But since i and j are summed over, it is also true that $B = S_{ji}A_{ji}$. Therefore, $B = -B$, implying that $B = 0$.

(d) The antisymmetric tensor A_{ij} has only three independent components (since the diagonal terms are zero), and therefore they can be expressed in terms of the three components of a vector ω_k. The only stipulation is that $A_{ij} = -A_{ji}$, and this is satisfied if:

$$A_{ij} = \epsilon_{ijk}\omega_k \rightarrow A_{ji} = \epsilon_{jik}\omega_k = -A_{ij} \tag{21}$$

The product $\epsilon_{ijk}A_{jk}$ is given by:

$$\begin{aligned} \epsilon_{ijk}A_{jk} &= \epsilon_{ijk}\epsilon_{jkl}\omega_l \\ &= (\delta_{il}\delta_{jj} - \delta_{ij}\delta_{lj})\omega_l \\ &= 2\omega_i \end{aligned} \tag{22}$$

4.

$$F_{ij} \int_V dV\, f(r)\, r_i r_j \tag{23}$$

First note that the only vector F_{ij} can depend on is a_i. So F_{ij} has to be a product of a_i, δ_{ij} and ϵ_{ijk}. The only permissible combination of these is:

$$F_{ij} = A\delta_{ij} + Ba_i a_j \tag{24}$$

For evaluating A and B, we need two equations. The first of these can be obtained using:

$$\delta_{ij}F_{ij} = 3A + B = I_1 \tag{25}$$

where the integral I_1 is:

$$\begin{aligned} I_1 &= \int_V dV\ f(r)\, r^2 \\ &= \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\theta_0} \sin(\theta) d\theta \int_{r=0}^{1} dr\ r^4 f(r) \\ &= 2\pi(1-\cos(\theta_0)) \int_0^1 dr\ f(r)\, r^4 \end{aligned} \tag{26}$$

The second equation is obtained from:

$$a_i a_j F_{ij} = A + B = I_2 \tag{27}$$

where the integral I_2 is:

$$\begin{aligned} I_2 &= \int_V dV\ f(r)(r_i a_i)^2 \\ &= \int_V dV\ f(r)\, r^2 \cos(\theta)^2 \\ &= \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\theta_0} \cos(\theta)^2 \sin(\theta) d\theta \int_{r=0}^{1} dr\ r^4 f(r) \\ &= (2\pi/3)(1-\cos(\theta_0)^3) \int_0^1 dr\ r^4\ f(r) \end{aligned} \tag{28}$$

The constants A and B are then given by:

$$A = (I_1 - I_2)/2 \tag{29}$$

$$B = (3I_2/2) - (I_1/2) \tag{30}$$

For a sphere, we would expect the result to be an isotropic tensor, and therefore $B = 0$. Since $\cos(\theta_0) = -1$ for a sphere, we find that:

$$I_1 = 4\pi \int_0^1 dr\ r^4\ f(r) \tag{31}$$

$$I_2 = (4\pi/3) \int_0^1 dr\ r^4\ f(r) \tag{32}$$

Using this, we get:

$$A = \frac{4\pi}{3} \int_0^1 dr\ r^4\ f(r) \tag{33}$$

$$B = 0 \tag{34}$$

5. The integral I_{ij} is:

$$\begin{aligned} I_{ij} &= \int_S dS a_i b_j a_k b_l x_k x_l \\ &= A\delta_{ij} + B a_i a_j + C b_i b_j + D a_i b_j + E a_j b_i \end{aligned} \tag{35}$$

Multiplying the above equation by δ_{ij}, $a_i a_j$ and $b_i b_j$ we get the following simultaneous equations for A, B and C:

$$3A + B + C = 0$$

$$A + B = 0$$

$$A + C = 0$$

These three can be solved to give $A = 0$, $B = 0$ and $C = 0$. Multiplying $\mathring{1}$ by $a_j b_i$ we get:

$$E = 0$$

Multiplying $\mathring{1}$ by $a_i b_j$ we get:

$$D = a_i^2 b_j^2 \int_S dS a_k b_l x_k x_l$$

In order to calculate this integral, take a and b along the x_1 and x_2 directions. Then the equation becomes:

$$D = \int_0^{2\pi} d\phi \int_0^{\pi} \sin(\theta) d\theta [\cos(\theta)\cos(\phi)\cos(\theta)\sin(\phi)]$$

It can easily be verified that the right side of the above equation is identically zero.

6. (a) Since we know that the unit vectors $\mathbf{e}_\xi$ and $\mathbf{e}_\eta$ are in the directions of $\nabla\xi$ and $\nabla\eta$ respectively, it is necessary to find these vectors. It is not possible to invert the expressions to determine ξ and η in terms of x and y, but we can find the expressions for the unit vectors directly. The unit vectors in the x and y directions are given by:

$$\begin{aligned} \nabla x = \mathbf{e}_x &= (\nabla\xi)\sinh(\xi)\cos(\eta) - (\nabla\eta)\cosh(\xi)\sin(\eta) \\ \nabla y = \mathbf{e}_y &= (\nabla\xi)\cosh(\xi)\sin(\eta) + (\nabla\eta)\sinh(\xi)\cos(\eta) \end{aligned} \tag{36}$$

The above simultaneous equations can be solved to determine $\nabla\xi$ and $\nabla\eta$:

$$\begin{aligned}\nabla\xi &= \frac{\sinh(\xi)\cos(\eta)\mathbf{e}_x + \cosh(\xi)\sin(\eta)\mathbf{e}_y}{[\sinh(\xi)\cos(\eta)]^2 + [\cosh(\xi)\sin(\eta)]^2} \\ \nabla\eta &= \frac{-\cosh(\xi)\sin(\eta)\mathbf{e}_x + \sinh(\xi)\cos(\eta)\mathbf{e}_y}{[\sinh(\xi)\cos(\eta)]^2 + [\cosh(\xi)\sin(\eta)]^2}\end{aligned} \tag{37}$$

The magnitude of the gradients $\nabla\xi$ and $\nabla\eta$ are:

$$|\nabla\xi| = |\nabla\eta| = \{[\sinh(\xi)\cos(\eta)]^2 + [\cosh(\xi)\sin(\eta)]^2\}^{-1/2} \tag{38}$$

and therefore the unit vectors $\mathbf{e}_\xi$ and $\mathbf{e}_\eta$ are:

$$\begin{aligned}\mathbf{e}_\xi &= \frac{\sinh(\xi)\cos(\eta)\mathbf{e}_x + \cosh(\xi)\sin(\eta)\mathbf{e}_y}{\{[\sinh(\xi)\cos(\eta)]^2 + [\cosh(\xi)\sin(\eta)]^2\}^{1/2}} \\ \mathbf{e}_\eta &= \frac{-\cosh(\xi)\sin(\eta)\mathbf{e}_x + \sinh(\xi)\cos(\eta)\mathbf{e}_y}{\{[\sinh(\xi)\cos(\eta)]^2 + [\cosh(\xi)\sin(\eta)]^2\}^{1/2}}\end{aligned} \tag{39}$$

(b) From the above equation, it can easily be verified that $\mathbf{e}_\xi \cdot \mathbf{e}_\eta = 0$, indicating that the coordinate system is an orthogonal one.

(c) The expressions for h_ξ and h_η can be derived using:

$$d\mathbf{x} = dx\mathbf{e}_x + dy\mathbf{e}_y \tag{40}$$

From equation 1̊, dx and dy are given by:

$$\begin{aligned}dx &= \sinh(\xi)\cos(\eta)d\xi - \cosh(\xi)\sin(\eta)d\eta \\ dy &= \cosh(\xi)\sin(\eta)d\xi + \sinh(\xi)\cos(\eta)d\eta\end{aligned} \tag{41}$$

In addition, the equation for $\mathbf{e}_x$ and $\mathbf{e}_y$ are given in terms of $\mathbf{e}_\xi$ and $\mathbf{e}_\eta$ in 1̊a. Using these and after some algebraic simplification, we get the following expression for $d\mathbf{x}$:

$$d\mathbf{x} = \{[\sinh(\xi)\cos(\eta)]^2 + [\cosh(\xi)\sin(\eta)]^2\}^{1/2}\{d\xi\mathbf{e}_\xi + d\eta\mathbf{e}_\eta\} \tag{42}$$

Therefore, we find that

$$h_\xi = h_\eta = \{[\sinh(\xi)\cos(\eta)]^2 + [\cosh(\xi)\sin(\eta)]^2\}^{1/2} \tag{43}$$

The Laplacian in an orthogonal coordinate system in two dimensions, using the extension of the formula derived in class is:

$$\nabla^2\phi = \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial y_1}\left(\frac{h_2}{h_1}\frac{\partial\phi}{\partial y_1}\right) + \frac{\partial}{\partial y_2}\left(\frac{h_1}{h_2}\frac{\partial\phi}{\partial y_2}\right)\right] \tag{44}$$

Using this formula, we find that:

$$\nabla^2\phi = \frac{1}{[\sinh(\xi)\cos(\eta)]^2 + [\cosh(\xi)\sin(\eta)]^2}\left(\frac{\partial^2\phi}{\partial\xi^2} + \frac{\partial^2\phi}{\partial\eta^2}\right) \tag{45}$$

7. The definition of $\nabla^2\phi$ is $\nabla\cdot(\nabla\phi)$ where $\nabla\phi$ is:

$$\nabla\phi = \mathbf{e_1}\frac{1}{h_1}\frac{\partial\phi}{\partial y_1} + \mathbf{e_2}\frac{1}{h_2}\frac{\partial\phi}{\partial y_2} + \mathbf{e_3}\frac{1}{h_3}\frac{\partial\phi}{\partial y_3} \tag{46}$$

Using the equation for the divergence of a vector derived in class,

$$\nabla\cdot(\nabla\phi) = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial y_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial\phi}{\partial y_1}\right) + \frac{\partial}{\partial y_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial\phi}{\partial y_2}\right) + \frac{\partial}{\partial y_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial\phi}{\partial y_3}\right)\right] \tag{47}$$

8 Kinematics

1.

2. The velocity field is,

$$v_z = V\left(1 - \frac{x^2+y^2}{R^2}\right) \tag{48}$$

Therefore, the rate of deformation tensor is given by

$$\nabla\mathbf{v} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (-2x/R) & (-2y/R) & 0 \end{pmatrix} \tag{49}$$

The isotropic part of this rate of deformation field is zero, while the symmetric and anti-symmetric parts are,

$$\mathbf{S} = \begin{pmatrix} 0 & 0 & (-x/R) \\ 0 & 0 & (-y/R) \\ (-x/R) & (-y/R) & 0 \end{pmatrix} \tag{50}$$

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & (x/R) \\ 0 & 0 & (y/R) \\ (-x/R) & (-y/R) & 0 \end{pmatrix} \tag{51}$$

3. The velocity field is given by $v_\theta = (\Omega/r)$. In cylindrical co-ordinates, it can easily be seen that the rate of deformation tensor is,

$$\nabla\mathbf{v} = \begin{pmatrix} (\partial v_r/\partial r) & (\partial v_\theta/\partial r) \\ (1/r)(\partial v_r/\partial\theta) - (v_\theta/r) & (v_r/r) + (1/r)(\partial v_\theta/\partial\theta) \end{pmatrix} \tag{52}$$

$$\nabla\mathbf{v} = \begin{pmatrix} 0 & -(\Omega/r^2) \\ -(\Omega/r^2) & 0 \end{pmatrix} \tag{53}$$

The stress tensor is symmetric in this co-ordinate system, and the curl of the velocity is zero. The velocities in a Cartesian co-ordinate system are,

$$v_x = -v_\theta \sin(\theta) = -\frac{\Omega y}{x^2 + y^2} \tag{54}$$

$$v_y = v_\theta \cos(\theta) = \frac{\Omega x}{x^2 + y^2} \tag{55}$$

The rate of deformation tensor in a Cartesian co-ordinate system is

$$\nabla\mathbf{v} = \begin{pmatrix} (2\Omega xy)/(x^2+y^2)^2 & \Omega(y^2-x^2)/(x^2+y^2)^2 \\ \Omega(y^2-x^2)/(x^2+y^2)^2 & (-2\Omega xy)/(x^2+y^2)^2 \end{pmatrix} \tag{56}$$

The stress tensor is symmetric. The vorticity can be verified to be zero in both the Cartesian and cylindrical co-ordinates.

4.

9 Conservation equations:

1.

$$\begin{aligned} \mathcal{D} &= 2\mu s_{ij}s_{ji} - (2/3)\mu s_{kk}^2 \\ &= 2\mu(s_{11}^2 + s_{22}^2 + s_{33}^2 + 2s_{12}s_{21} + 2s_{13}s_{31} + 2s_{23}s_{32}) \\ &\quad -(2/3)\mu(s_{11}^2 + s_{22}^2 + s_{33}^2 + 2s_{11}s_{22} + 2s_{11}s_{33} + 2s_{22}s_{33}) \\ &= 2\mu(s_{12}^2 + s_{13}^2 + s_{23}^2) + (2\mu/3)((s_{11}-s_{22})^2 + (s_{22}-s_{33})^2 + (s_{11}-s_{33})^2) \end{aligned} \tag{57}$$

10 Viscous flows:

1. (a) The problem can be taken as the superposition of two problems — one with the force of gravity parallel to the line joining the sphere's centeres, and the other with the force of gravity perpendicular to the line joining the centers. It is clear from reversibility that if two spheres separated when they fell, they would come together when the direction of gravity is reversed. This violates symmetry, so the two spheres can neither come together not separate from each other, and therefore the distance between their line of centers remains a constant.

   (b) When $\mathbf{F}_g$, the force due to gravity, is to $\mathbf{p}$, $\mathbf{U} = U_{\|} g_i$. When $\mathbf{F}_g$ is $\perp$ to $\mathbf{p}$, then $\mathbf{U} = \frac{1}{2} U_{\|} g_i$. In the general case where $\mathbf{F}_g$ is at an angle to $\mathbf{p}$, then the components in the $\|$ and $\perp$ directions are:

$$F_{\|} = F_g \cos(\theta) \quad F_{\perp} = F_g \sin(\theta) \tag{58}$$

   This induces a velocity $U_{\|} \cos(\theta)$ parallel to $\mathbf{p}$ and a velocity $\frac{1}{2} U_{\|} \sin(\theta)$ perpendicular to $\mathbf{p}$. Therefore, the velocity in the vertical direction is given by:

$$\begin{aligned} U_v &= (U_{\|} \cos(\theta)) \cos(\theta) + \left(\tfrac{1}{2} U_{\|} \sin(\theta)\right) \sin(\theta) \\ &= \tfrac{1}{2} U_{\|} (1 + \cos(\theta)^2) \end{aligned} \tag{59}$$

   The velocity in the horizontal direction is:

$$\begin{aligned} U_h &= (U_{\|} \cos(\theta)) \sin(\theta) - (\tfrac{1}{2} U_{\|} \sin(\theta)) \cos(\theta) \\ &= \tfrac{1}{2} U_{\|} \cos(\theta) \sin(\theta) \end{aligned} \tag{60}$$

2.

3. It is convenient to separate the fluid velocity $u_i = G_{ij} x_j + u'_i$, where $u'_i \to 0$ for $r \to \infty$. The velocity u'_i is obtained by solving the equation:

$$\mu \nabla^2 u'_i = \nabla p \quad \nabla^2 p = 0 \tag{61}$$

   The solution of this is given by:

$$u'_i = u'_{ih} + (1/2\mu) x_i p \tag{62}$$

where u'_{ih} is the solution of the homogeneous equation $\nabla^2 u'_{ih} = 0$.

The velocity u'_i and the pressure p can depend only on the tensor G_{ij}, and so the possible solutions are:

$$p = 2\mu\lambda_1 G_{jk}\left(\frac{\delta_{jk}}{r^3} - \frac{3x_j x_k}{r^5}\right) \tag{63}$$

$$u'_{ih} = \frac{\lambda_2 G_{ij} x_j}{r} + \lambda_3 G_{jk}\left(\frac{15 x_i x_j x_k}{r^7} - \frac{3(\delta_{ij}x_k + \delta_{jk}x_i + \delta_{ik}x_j)}{r^5}\right) \tag{64}$$

The equation for the pressure is inserted into $\mathring{2}$, and we can simplify the above expressions by noting that $G_{ii} = 0$:

$$u'_i = \lambda_1 G_{jk}\left(\frac{-3x_i x_j x_k}{r^5}\right) + \lambda_2\left(\frac{G_{ij}x_j}{r^2}\right) + \lambda_3 G_{jk}\left(\frac{15 x_i x_j x_k}{r^7} - \frac{3(x_j\delta_{ik} + x_k\delta_{ij})}{r^5}\right) \tag{65}$$

One of the constants in the above equation can be determined from the equation of continuity:

$$\partial_i u'_i = 0 \tag{66}$$

$$\begin{aligned}\lambda_1 &\left(\frac{-3(\delta_{ii}x_j x_k + \delta_{ij}x_i x_k + \delta_{ik}x_i x_j)}{r^5} + \frac{15 x_i x_i x_j x_k}{r^7}\right) G_{jk} \\ &+ \lambda_2 G_{ij}\left(\frac{\delta_{ij}}{r^3} - \frac{3x_i x_j}{r^5}\right) \\ &+ \lambda_3 G_{jk}\left(\frac{15(\delta_{ii}x_j x_k + \delta_{ij}x_i x_k + \delta_{ik}x_i x_j)}{r^7} - \frac{105 x_i x_i x_j x_k}{r^9}\right. \\ &\left. - \frac{3(\delta_{ij}\delta_{ik} + \delta_{ik}\delta_{ij})}{r^5} + \frac{15(x_i x_j \delta_{ik} + x_i x_k \delta_{ij})}{r^7}\right) = 0\end{aligned} \tag{67}$$

The above equation can be simplified using the conditions that $G_{ii} = 0$ and $x_i x_i = r^2$.

$$-3\lambda_2 G_{ij}\frac{x_i x_j}{r^5} = 0 \rightarrow \lambda_2 = 0 \tag{68}$$

In addition, the center of the particle is at the origin and is stationary, so we have the condition:

$$u_i = 0 \text{ at } r = a \tag{69}$$

This gives us:

$$\lambda_1 G_{jk}\left(\frac{-3x_i x_j x_k}{r^5}\right) + \lambda_3 G_{jk}\left(\frac{15 x_i x_j x_k}{a^7} - \frac{6x_k}{a^6}\right) + G_{ij}x_i = 0 \tag{70}$$

The above equation can be easily solved to give:

$$\lambda_1 = \frac{5a^3}{6} \quad \lambda_3 = \frac{a^5}{5} \tag{71}$$

Inserting these into the expression for the velocity $\mathring{5}$, the final expression for the velocity is:

$$\begin{aligned} u_i = & \quad G_{ij}x_j - \frac{5}{2}\frac{a^3x_ix_jx_kG_{jk}}{r^5} + G_{jk}\left(\frac{5}{2}\frac{a^5x_ix_jx_k}{r^7} - \frac{a^5(x_j\delta_{ik} + x_k\delta_{ij})}{2r^5}\right) \\ p = & \quad -\frac{5\mu a^3G_{jk}x_jx_k}{r^5} \end{aligned} \tag{72}$$

The stress acting on the particle is given by:

$$T_{il} = -p\delta_{il} + \mu\left(\frac{\partial u_i}{\partial x_l} + \frac{\partial u_l}{\partial x_i}\right) \tag{73}$$

At the surface of the particle, $(\partial u_i/\partial x_l)$ is:

$$\begin{aligned} \frac{\partial u_i}{\partial x_l} = & \quad G_{il} - \frac{5}{2a^2}G_{jk}(\delta_{il}x_jx_k + \delta_{jl}x_ix_k + \delta_{kl}x_ix_j) + \frac{25}{2a^4}G_{jk}(x_ix_jx_kx_l) \\ & + \frac{5}{2a^2}G_{jk}(\delta_{il}x_jx_k + \delta_{jl}x_ix_k + \delta_{kl}x_ix_j) - \frac{35}{2a^4}x_ix_jx_kx_lG_{jk} \\ & - \frac{1}{2}(\delta_{jl}\delta_{ik} + \delta_{kl}\delta_{ij})G_{jk} + \frac{5}{2a^2}G_{jk}(x_jx_l\delta_{ik} + x_kx_l\delta_{ij}) \\ = & \quad \frac{-5}{a^4}x_ix_jx_kx_lG_{jk} + \frac{5}{a^2}G_{ik}x_kx_l \end{aligned} \tag{74}$$

Using the above relation, the stress is:

$$T_{il} = \frac{5\mu}{a^2}(\delta_{il}x_jx_k + \delta_{ij}x_kx_l + \delta_{jl}x_ix_k)G_{jk}) - \frac{10\mu}{a^4}(x_ix_jx_kx_lG_{jk}) \tag{75}$$

The product $T_{il}n_lx_m$ is given by:

$$\begin{aligned} T_{il}n_lx_m & = T_{il}x_m\frac{x_l}{a} \\ & = \frac{5\mu}{a^3}(2x_ix_jx_kx_m + \delta_{ij}x_kx_l^2x_m)G_{jk} - \frac{10\mu}{a^4}(x_ix_jx_kx_l^2x_m)G_{jk} \\ & = \frac{5\mu}{a^3}G_{ik}x_kx_m \end{aligned} \tag{76}$$

The integral of this over the surface of a aphere is:

$$\int_A dA\, \frac{5\mu}{a^3} G_{ik} x_k x_m = 5\mu G_{im} \left(\frac{4}{3} \pi a^3 \right) \tag{77}$$

4. (a) At low Reynolds number, the velocity can be separated into a homogeneous and a particular solution:

$$u_i = u_{hi} + u_{pi} \tag{78}$$

The homogeneous solution is obtained from the equation:

$$\partial_j^2 u_{hi} = 0 \tag{79}$$

while the particular solution can be expressed as a function of the pressure:

$$u_{pi} = \frac{1}{2\mu} p x_i \tag{80}$$

where the pressure is obtained by solving:

$$\partial_j^2 p = 0 \tag{81}$$

Note that the angular velocity vector Ω is a pseudo vector, while u_i and p are real vectors. Since Ω is the only vector in the system, u_i and p are linear functions of Ω. However, there is no way to make a real scalar p from δ_{ij}, ϵ_{ijk} and the pseudo vector Ω_i. Therefore,

$$p = 0 \tag{82}$$

Therefore, the particular solution for the velocity is also zero. The general solution for the velocity has to be of the form:

$$u_i = \frac{A}{r} + \frac{B_i x_i}{r^3} + \dots \tag{83}$$

The only way to make a real vector u_i which decays as $r \to \infty$ and is linear in Ω_k is:

$$u_i = \lambda \epsilon_{ijk} \Omega \frac{x_k}{r^3} \tag{84}$$

where λ is a constant. Note that the incompressibility condition is automatically satisfied by the above equation:

$$\partial_i u_i = \lambda \epsilon_{ijk} \Omega_j \left(\frac{\delta_{ik}}{r^3} - \frac{3x_i x_k}{r^5} \right) = 0 \tag{85}$$

The constant λ can be determined from the boundary condition that at $r = a$, $u_i = \epsilon_{ijk}\Omega_j x_k$. This gives $\lambda = a^3$, and therefore the solution for the velocity is:

$$u_i = \epsilon_{ijk} \Omega_j x_k a^3 / r^3 \tag{86}$$

(b) The force per unit area on the surface is $f_i = T_{il} n_l$. Since the pressure is zero, the stress tensor T_{il} is given by:

$$T_{il} = \mu(\partial_l u_i + \partial_i u_l) \tag{87}$$

The strain rate is:

$$\begin{aligned} \partial_l u_i &= \epsilon_{ijk} \Omega_j a^3 \left(\frac{\delta_{kl}}{r^3} - \frac{3x_k x_l}{r^5} \right) \\ \partial_i u_l &= \epsilon_{ljk} \Omega_j a^3 \left(\frac{\delta_{ik}}{r^3} - \frac{3x_i x_k}{r^5} \right) \end{aligned} \tag{88}$$

Therefore, the force f_i is given by:

$$\begin{aligned} f_i &= T_{il} \frac{x_l}{r} \\ &= \mu a^3 \left[\epsilon_{ijk} \Omega_j \left(\frac{\delta_{kl} x_l}{r^4} - \frac{3x_k x_l x_l}{r^6} \right) \right. \\ &\left. + \epsilon_{ljk} \Omega_j a^3 \left(\frac{\delta_{ik} x_l}{r^4} - \frac{3x_i x_k x_l}{r^5} \right) \right] \end{aligned} \tag{89}$$

The fourth term on the right side of the above equation contains $\mathbf{x} \times \mathbf{x} = 0$. Further, it can be easily verified that the first and third terms cancel, and so the force is given by:

$$f_i = \frac{-3\mu a^3 \epsilon_{ijk} \Omega_j x_k}{r^4} \tag{90}$$

The torque on the sphere is:

$$\begin{aligned} L_m &= \int dA \epsilon_{min} x_j (-3\mu/a) \epsilon_{ijk} \Omega_j x_k \\ &= \epsilon_{min} \epsilon_{ijk} \Omega_k (-3\mu/a) \int dA x_n x_k \end{aligned} \tag{91}$$

In order to calculate the integral in the above equation, use directional symmetries:

$$\begin{aligned} \int dA x_n x_k &= \lambda \delta_{nk} \\ \int dA r^2 &= 3\lambda \\ \lambda &= \frac{4\pi a^4}{3} \end{aligned} \tag{92}$$

Therefore, we get

$$\begin{aligned} L_m &= \epsilon_{min}\epsilon_{ijk}\Omega_j(-4\pi\mu a^3) \\ &= -8\pi\mu a^3\Omega_m \end{aligned} \tag{93}$$

5. Consider a cylindrical coordinate system with the origin located at the wall perpendicular to the center of the disk. It is appropriate to non - dimensionalise the length, time and velocity scales as follows:

$$z = \frac{z^*}{a\epsilon} \quad r = \frac{r^*}{a} \quad u_z = \frac{u_z^*}{U} \quad u_r = \frac{u_r^*\epsilon}{U} \quad t = \frac{t^*\epsilon U}{a} \tag{94}$$

where the variables with the superscript * are dimensional, while those without the superscript are dimensional. The scaling for u_r in the above equation was determined from the mass conservation equation:

$$\frac{1}{r^*}\frac{\partial(r^* u_r^*)}{\partial r^*} + \frac{\partial u_z^*}{\partial z^*} = 0 \tag{95}$$

With the above scaling, the momentum equations in the r direction is:

$$\frac{Re}{\epsilon^2}\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + u_z\frac{\partial u_r}{\partial z}\right) = -\frac{a}{\mu U}\frac{\partial p^*}{\partial r} + \left(\frac{1}{\epsilon^3}\frac{\partial^2 u_r}{\partial z^2} + \frac{1}{\epsilon}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right) - \frac{1}{\epsilon}\frac{u_r^2}{r}\right) \tag{96}$$

In the leading approximation, this equation reduces to:

$$\frac{\partial^2 u_r}{\partial z^2} = \frac{\epsilon^3 a}{\mu U}\frac{\partial p^*}{\partial r} + O(Re\epsilon) + O(\epsilon^2) \tag{97}$$

From this equation, it is appropriate to scale p^* by $(\mu U/a\epsilon^3)$, and the leading order equation is:

$$\frac{\partial^2 u_r}{\partial z^2} = \frac{\partial p}{\partial r} \tag{98}$$

The momentum conservation equation in the z direction is:

$$\frac{Re}{\epsilon}\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+u_z\frac{\partial u_z}{\partial z}\right)=-\frac{1}{\epsilon^4}\frac{\partial p}{\partial z}+\frac{1}{\epsilon^2}\frac{\partial^2 u_z}{\partial z^2}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)-\frac{u_z}{r^2} \tag{99}$$

In the limit $\epsilon \to 0$, this equation is:

$$\frac{\partial p}{\partial z}=0+O(\epsilon^2)+O(\epsilon^3 Re) \tag{100}$$

The boundary conditions are:

$$\begin{array}{llll} u_r=0 & u_z=0 & \text{at} & z=0 \\ u_r=0 & u_z=-1 & \text{at} & z=1 \end{array} \tag{101}$$

Equation $\mathring{7}$ for the pressure field implies that p is only a function of r ($p=p(r)$). Using this information, equation $\mathring{5}$ for the radial velocity can be solved:

$$u_r=\frac{1}{2}\frac{\partial p}{\partial r}(z^2-z) \tag{102}$$

This solution satisfies the boundary condition $\mathring{8}$ $u_r=0$ at $z=0$ and $z=1$. The value of the pressure gradient can be obtained by examining the continuity equation:

$$\frac{\partial u_z}{\partial z}=-\frac{1}{r}\frac{\partial(ru_r)}{\partial r} \tag{103}$$

Integrating this between $z=0$ and 1, we get

$$\begin{aligned} u_z|_{z=0}^{1} &= -\int_0^1 dz\frac{1}{r}\frac{\partial(ru_r)}{\partial r} \\ -1 &= \frac{-1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)\left(\frac{z^3}{6}-\frac{z^2}{4}\right)_{z=0}^{1} \\ -1 &= \frac{1}{12}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) \end{aligned} \tag{104}$$

The above equation for the pressure can be solved to obtain:

$$p=-3r^2+C_1\log(r)+C_2 \tag{105}$$

The constant C_1 is zero because the pressure has to be finite at $r = 0$, while $C_2 = 3$ because $p = 0$ at $r = 1$. Therefore, the pressure and velocity are:

$$p = 3(1 - r^2) \quad u_r = 3r(z - z^2) \quad u_z = (2z^3 - 3z^2) \tag{106}$$

The force on the disk per unit area is:

$$f_z = -n_z T_{ij} n_z = p^* - 2\mu \frac{\partial u_z^*}{\partial z^*} \tag{107}$$

There is a negative term in the above equation because the unit normal to the disk is in the $-z$ direction. The second term on the right side of the above equation can be neglected, because it is $O(\epsilon^2)$ smaller than the first term. Therefore, the force per unit area is:

$$f_z = \frac{\mu U}{a\epsilon^3} 3(1 - r^2) \tag{108}$$

The total force is determined by integrating the force over the area of the disk:

$$\begin{aligned} F_z &= \int_0^{2\pi} d\theta \int_0^a dr^* \, r^* \left(\frac{\mu U}{a\epsilon^3} 3 \left(1 - \frac{r^{*2}}{a^2} \right) \right) \\ &= \frac{-3\pi\mu a U}{2\epsilon^3} \end{aligned} \tag{109}$$

6. (a) It is appropriate to choose a two dimensional Cartesian coordinate system with the origin on the channel wall. The equation for the surface of the cylinder is:

$$(x_{c2}^* - x_2^*)^2 + x_1^{*2} = a^2 \tag{110}$$

where * is used to denote dimensional quantities, and $x_{c2}^* = a(1+\epsilon)$ is the position of the center of the cylinder. The above equation can be reduced to:

$$x_2^* = a(1 + \epsilon) - \sqrt{a^2 + x_1^{*2}} \tag{111}$$

The coordinate x_2^* is scaled by ϵa, and the coordinate x_1^* by $\epsilon^{1/2} a$ to obtain the following equation for the surface:

$$x_2 = H(x_1) = 1 + \frac{x_1^2}{2} + \ldots \tag{112}$$

The Navier - Stokes mass and momentum equations are:

$$\partial_1^* u_1^* + \partial_2^* u_2^* = 0 \tag{113}$$

$$\rho(\partial_t^* u_1^* + u_1^* \partial_1^* u_1^* + u_2^* \partial_2^* u_1^*) = -\partial_1^* p^* + \mu(\partial_1^{*2} + \partial_2^{*2}) u_1^* \tag{114}$$

$$\rho(\partial_t^* u_2^* + u_1^* \partial_1^* u_2^* + u_2^* \partial_2^* u_2^*) = -\partial_2^* p^* + \mu(\partial_1^{*2} + \partial_2^{*2}) u_2^* \tag{115}$$

Scaling the velocity u_1^* by U, the velocity u_2^* by $\epsilon^{1/2}U$ and the pressure by $(\mu U/\epsilon^{3/2}a^2)$, the dimensionless equations are:

$$\partial_1 u_1 + \partial_2 u_2 = 0 \tag{116}$$

$$(\rho U a \epsilon^{3/2}/\mu)(\partial_t + u_1\partial_1 + u_2\partial_2)u_1 = -\partial_1 p + (\partial_2^2 + \epsilon\partial_1^2)u_1 \tag{117}$$

$$(\rho U a \epsilon^{5/2})(\partial_t + u_1\partial_1 + u_2\partial_2)u_2 = -\partial_2 p + \epsilon(\partial_2^2 + \epsilon\partial_1^2)u_2 \tag{118}$$

The inertial terms in the above equations can be neglected for $(\rho U a \epsilon^{3/2}/\mu) \ll 1$.

(b) The boundary conditions required for solving the above problem are:

$$\begin{array}{lll} u_1 = 0 & \text{at} & x_2 = 0 \\ u_2 = 0 & \text{at} & x_2 = 0 \\ u_1 = 1 & \text{at} & x_2 = H(x_1) \\ u_2 = 0 & \text{at} & x_2 = H(x_1) \end{array} \tag{119}$$

In addition, there should also be no net flow of fluid across any surface, so:

$$\int_0^{H(x_1)} dx_2 \, u_2 = 0 \tag{120}$$

(c) The Stokes equations are:

$$\partial_1 u_1 + \partial_2 u_2 = 0 \tag{121}$$

$$-\partial_1 p + \partial_2^2 u_1 = 0 \tag{122}$$

$$-\partial_2 p = 0 \tag{123}$$

The two momentum equations can be easily solved along with the boundary conditions and the zero net flux condition to obtain:

$$u_1 = \left(\frac{3x_2^2}{H^2} - \frac{2x_2}{H}\right) \tag{124}$$

$$\partial_1 p = \frac{6}{H^2} \tag{125}$$

11 Potential flow:

1. The equation for the potential for a cylinder moving with a constant velocity U_i is:

$$\phi = \lambda \frac{U_i x_i}{r^2} \tag{126}$$

and the fluid velocity is:

$$u_i = \lambda U_j \left(\frac{\delta_{ij}}{r^2} - \frac{2 x_i x_j}{r^4} \right) \tag{127}$$

The constant λ can be determined from the condition that $u_i n_i = U_i n_i$ at the surface of the cylinder $r = a$:

$$\lambda U_j \left(\frac{x_j}{a^3} - \frac{2 x_i^2 x_j}{a^5} \right) = \frac{U_i x_i}{a} \tag{128}$$

This can be solved to obtain $\lambda = -(1/a^2)$. Therefore, the equation for the potential is:

$$\phi = -\frac{U_i x_i a^2}{r^2} \tag{129}$$

The pressure field is given by:

$$p = p_0 - \rho \frac{\partial \phi}{\partial t} - \rho U_i u_i - \frac{\rho u_i^2}{2} \tag{130}$$

The first, third and fourth terms in the above equation do not contribute to the net force on the cylinder, and the only contribution is from the second term which is given by:

$$F_i = \int dA p n_i \tag{131}$$

The force along the direction of flow is:

$$\begin{aligned} F_i &= \frac{dU_j}{dt} \int dA \frac{a^2 x_i x_j}{r^3} \\ &= \frac{dU_j}{dt} \left(\rho \pi a^2 \right) \end{aligned} \tag{132}$$

Therefore, the added mass of the cylinder is equal to the mass of fluid displaced by it.

2. The equation for the stream function in the presence of a source at $(-d/2)$ and a sink at $(d/2)$ and a uniform flow in the x direction is:

$$\psi = \frac{m\theta_1}{2\pi} + \frac{-m\theta_2}{2\pi} + Ur\sin(\theta) \tag{133}$$

where θ_1 is the angle subtended at the source and θ_2 is the angle subtended at the sink. The angles θ_1 and θ_2 can be related to θ and r as follows:

$$\tan(\theta_1) = \frac{r\sin(\theta)}{(d/2) + r\cos(\theta)} \quad \tan(\theta_2) = \frac{r\sin(\theta)}{r\cos(\theta) - (d/2)} \tag{134}$$

These can be inserted into the above equation, and we can set $\psi = 0$ to obtain the equation for the surface of the body:

$$\left[\tan^{-1}\left(\frac{r\sin(\theta)}{r\cos(\theta) - (d/2)}\right) - \tan^{-1}\left(\frac{r\sin(\theta)}{r\cos(\theta) + (d/2)}\right)\right] + \frac{2\pi Ur}{m}\sin(\theta) = 0 \tag{135}$$

In the limit $d \to 0$, the angles θ_1 and θ_2 are given by:

$$\theta_1 = \theta - \frac{d\sin(\theta)}{2r} \quad \theta_2 = \theta + \frac{d\sin(\theta)}{2r} \tag{136}$$

Therefore, the equation for the streamline $\psi = 0$ becomes:

$$\frac{d\sin(\theta)}{r} = \frac{2\pi Ur\sin(\theta)}{m} \tag{137}$$

which is the equation for an infinite cylinder of radius:

$$r = \left(\frac{md}{2\pi U}\right)^{1/2} \tag{138}$$

3. (a) The fluid velocity field due to a sphere moving in potential flow is:

$$u_i = \frac{-U_j a^3}{2}\left(\frac{\delta_{ij}}{r^3} - \frac{3x_i x_j}{r^5}\right) \tag{139}$$

The kinetic energy per unit mass is given by:

$$\begin{aligned} \frac{\rho u_i^2}{2} &= \frac{\rho U_j U_k a^6}{8}\left(\frac{\delta_{ij}}{r^3} - \frac{3x_i x_j}{r^5}\right)\left(\frac{\delta_{ik}}{r^3} - \frac{3x_i x_k}{r^5}\right) \\ &= \left(\frac{\rho U_i^2 a^6}{8r^6} + \frac{3\rho U_j U_k a^6 x_j x_k}{8r^8}\right) \\ &= \frac{\rho U^2 a^6}{8r^6}(1 + 3\cos(\theta)^2) \end{aligned} \tag{140}$$

where θ is the angle made by the position vector with the direction of the velocity vector.

The total kinetic energy is can be easily calculated from the above expression:

$$\begin{aligned}\int_V dV \frac{\rho u_i^2}{2} &= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin(\theta) \int_a^{\infty} dr\, r^2 \frac{\rho U^2 a^6}{8 r^6}(1 + 3\cos(\theta)^2) \\ &= \frac{\pi \rho U^2 a^3}{3}\end{aligned} \tag{141}$$

The added mass obtained is given by:

$$M = \frac{KE}{(U^2/2)} = \frac{2\pi a^3 \rho}{3} \tag{142}$$

This is exactly half the mass of the fluid displaced by the sphere.

(b) The fluid velocity field for a viscous flow is given by:

$$u_i = U_j \left[\frac{3a}{4r} \left(\delta_{ij} + \frac{x_i x_j}{r^2} \right) + \frac{a^3}{r^3} \left(\delta_{ij} - \frac{3 x_i x_j}{r^2} \right) \right] \tag{143}$$

Since the velocity decays as (a/r) and the volume increases as r^3, the total kinetic energy is $O(\rho U^2 a^2 r)$, which becomes infinite for an infinite volume. The situation can be rectified, however, by realising that the Stokes flow approximation is valid only for $(r/a) \ll (1/Re)$, and beyond this the Oseen approximation has to be used. This decays much faster. Therefore, we can estimate the kinetic energy as $O(\rho U^2 a^3 / Re)$. This is much larger than the kinetic energy of $O(\rho U^2 a^3)$ in potential flow for $Re \ll 1$. This might be expected from the minimum energy theorem, which states that the potential flow has energy which is small compared to any other flow.

4. (a) The condition for the flow to be irrotational is that the circulation along any streamline should be a constant. This requires that:

$$\Omega_1 R_1^2 = \Omega_2 R_2^2 \tag{144}$$

The fluid velocity as a function of radius is given by:

$$v_\theta = \frac{\Omega_1 R_1^2}{r} \tag{145}$$

(b) The momentum conservation equation for the fluid in the r and z directions if viscous forces are neglected at steady state is given by:

$$\partial_z p = \rho g \tag{146}$$

$$\partial_r p = \frac{v_\theta^2}{r} \tag{147}$$

These equations can be integrated to give:

$$(p_0/\rho) + gz + \frac{\Omega_1 R_1^2}{2r^2} = 0 \tag{148}$$

This gives the equation for the surface. This is also identical to the Bernoulli equation for the fluid at the surface. Therefore, we find that $z \sim (1/r^2)$ for this interface.

5. The coordinates in the z and z' planes are related by

$$x = x' + \frac{a^2 x'}{x'^2 + y'^2} \tag{149}$$

$$y = y' - \frac{a^2 y'}{x'^2 + y'^2} \tag{150}$$

A circle of radius a is transformed onto a line of length $2a$ along the x axis centered at the origin. A flow around a cylinder in the z' plane gets converted into a flow past a flat surface.

A circle of radius b in the z' plane gets converted onto an ellipse of manor and minor axes $(1 + a^2/b^2)$ and $(1 - a^2/b^2)$ in the z plane.

6.

7. The velocity field due to the potential flow around a sphere with velocity U_i is:

$$u_i = -\frac{U_j a^3}{2}\left(\frac{\delta_{ij}}{r^3} - \frac{3x_i x_j}{r^5}\right)$$

The rate of dissipation of energy due to viscous dissipation is:

$$D = \frac{1}{2}\eta \int_V dV (\partial_i u_j + \partial_j u_i)(\partial_i u_j + \partial_j u_i)$$

where η is the viscosity. Since the flow is irrotational, the strain tensor is symmetric and the equation for the dissipation rate becomes:

$$D = 2\eta \int_V dV (\partial_i u_j)(\partial_j u_i)$$

The equation for the strain rate is:

$$\partial_j u_i = -\frac{U_k a^3}{2} \left(\frac{-3(\delta_{ik} x_j + \delta_{ij} x_k + \delta_{jk} x_i)}{r^5} + \frac{15 x_i x_j x_k}{r^7} \right)$$

The product $(\partial_i u_j)(\partial_j u_i)$ can be easily calculated from the above relation:

$$(\partial_i u_j)(\partial_j u_i) = \frac{U_k U_l a^6}{4} \left(\frac{18\delta_{kl} x_i^2 + 36 x_k x_l}{r^{10}} \right)$$

The above equation can be easily integrated over the volume of the fluid in spherical coordinates (using $x_k x_l = r^2 \cos(\theta)^2$ and $x_i^2 = r^2$) to give:

$$D = 12\pi\eta a U^2$$

The drag force is given by:

$$F_D = (D/U) = 12\pi\eta a U$$

This is twice the drag force due at zero Reynolds number ($F_D = 6\pi\eta a U$). We would expect the drag in potential flow to be greater, due to the mimimum dissipation theorem which states that the energy dissipation in a zero Reynolds number flow is lower than that in any other flow.

8. The force acting on the object is:

$$F_i = -\int_S dS p n_i \tag{151}$$

where S is the surface of the sphere. From the Bernoulli equation, this is:

$$F_i = \int_S dS \rho \left(\frac{1}{2} u_j^2 - U_j u_j \right) n_i \tag{152}$$

The first term on the right can be simplified as follows:

$$\int_{S_\infty} dS_\infty \frac{1}{2} u_j^2 n_i + \int_{S_w} dS_w \frac{1}{2} u_j^2 n_i - \int_S dS \frac{1}{2} u_j^2 n_i = \int_V dV u_j \partial_i u_j \tag{153}$$

where S_w is the surface of the wall and S_∞ is the surface at infinity. The first term on the left is zero because the velocity decays as r^3 at large r for a body in steady motion. The right side of the above equation in an irrotational flow is:

$$\begin{aligned}\int_V dV u_j \partial_i u_j &= \int_V dV u_j \partial_j u_i \\ &= \int_{S_\infty} dS_\infty u_j n_j u_i + \int_{S_w} dS_w u_j n_j u_i - \int_S dS u_j n_j u_i\end{aligned} \tag{154}$$

Once again neglecting the contribution from the surface at infinity, and at the wall $u_j n_j$ is zero because there is no flux through the wall. Also, at the surface of the object, $u_j n_j = U_j n_j$. With this simplification, we get:

$$F_i = \rho \int_{S_w} dS_w \frac{1}{2} u_j^2 n_i + U_j \int_S dS(u_i n_j - u_j n_i) \tag{155}$$

It can be show, using methods similar to that used in the class, that

$$\int_S dS(u_j n_i - u_i n_j) = \int_{S_\infty} dS_{infty}(u_j n_i - u_i n_j) + \int_{S_w} dS_w(u_j n_i - u_i n_j) \tag{156}$$

The first integral on the right side is zero, and the final expression for the force on the object is:

$$F_i = \rho \int_{S_w} dS_w \left(\frac{1}{2} u_j^2 n_i + U_j(u_i n_j - u_j n_i)\right) \tag{157}$$

At the wall, the unit normal n_i is $(0, -1, 0)$. Using this, the forces in the two directions can easily be determined:

$$F_1 = 0 \tag{158}$$

$$F_2 = \int_{S_w} dS_w \left(\frac{1}{2} u_i^2 + U_1 u_1\right) \tag{159}$$

9. The fluid velocity due to surface displacements is $O(\omega\xi_0)$, while the length scale is the wavelength of the fluctuations λ. Therefore, the $u_j \partial_j u_i$ term is $O(\omega^2 \xi_0^2/\lambda)$, while the $\partial_t u_i$ term is $O(\omega^2 \xi_0)$. The former can be neglected compared to the latter for $(\xi_0/\lambda) \ll 1$, or when the amplitude of the fluctautions is small compared to the wavelength.

Consider a coordinate system where the z axis is in the vertical direction, and the fluid occupies the space $z \leq 0$ in the absence of fluctuations. The equation for the surface and the velocity potential can be expressed as a function of the wave number and frequency as follows:

$$\xi = \xi_0 \exp{(ikx + i\omega t)} \quad \phi = f(z) \exp{(ikx + i\omega t)}$$

The equation for the velocity potential is:

$$\partial_i^2 \phi = 0 \implies (\partial_z^2 - k^2) f(z) = 0$$

The above equation, along with the boundary condition that $\phi \to 0$ for $z \to -\infty$, implies that:

$$f = C_1 \exp{(kz)}$$

The constant C_1 is determined from the boundary condition that $u_z = \partial_t \xi$ at $z = 0$.

$$C_1 = k^{-1} i\omega \xi_0$$

The Bernoulli equation for the pressure at the surface is:

$$p + \rho(\partial_t \phi + g\xi) = p_0$$

where p_0 is the pressure above the surface. At equilibrium, in the absence of any flow, we have:

$$p = p_0$$

Therefore, the equation for the displacement field is:

$$\partial_t \phi + g\xi = 0$$

Inserting the expressions for ϕ and ξ, we find that the frequency is given by:

$$\omega = \sqrt{(gk)}$$

10. For the potential flow around the sphere, the velocity far from the sphere is given by $(\partial \phi / \partial x_i) = G_{ij} x_j$. Therefore, the potential is of the form,

$$\phi = \frac{G_{jk} x_j x_k}{2} + B G_{jk} \left(-\frac{\delta_{jk}}{r^3} + \frac{3 x_j x_k}{r^5} \right) \tag{160}$$

Since the flow is incompressible, $\delta_{jk}G_{kj} = 0$. At the surface, $u_i n_i = n_i(\partial\phi/\partial x_i) = 0$. Using these two conditions, we obtain,

$$\phi = \frac{G_{jk}x_j x_k}{2}\left(1 + \frac{r^3}{3R^3}\right) \tag{161}$$

12 Boundary layer theory:

1. For the outer potential flow solution, we have:

$$\rho^{-1}\partial_1 p = U_1\partial_1 U_1 + U_3\partial_3 U_1 = 0 \tag{162}$$

$$\rho^{-1}\partial_3 p = U_1\partial_1 U_3 + U_3\partial_3 U_3 = 0 \tag{163}$$

Therefore the pressure is a constant in the outer flow. It is also important to note that there is no variation in the x_3 direction for a plate that is infinite in that direction. The equations for the fluid velocity in the boundary layer are:

$$\partial_1 u_1 + \partial_2 u_2 = 0 \tag{164}$$

$$u_1\partial_1 u_1 + u_2\partial_2 u_1 = \nu\partial_2^2 u_1 \tag{165}$$

$$u_1\partial_1 u_3 + u_2\partial_2 u_3 = \nu\partial_2^2 u_3 \tag{166}$$

The mass conservation equation $\mathring{3}$ and the momentum equation in the x_1 direction $\mathring{4}$ are identical to those for the Blasius flow over a flat plate, and the solution for the fluid velocity profile is identical to that obtained in class.

$$u_1 = U_1 f'(\eta) \tag{167}$$

where $\eta = x_2/\sqrt{(\nu x_1/U_1)}$, and $f(\eta)$ is the solution of the equation:

$$f''' + (1/2)ff'' = 0 \tag{168}$$

The equation for the velocity u_3 $\mathring{5}$ is identical to that for u_1, therefore the solution for u_3 will be proportional to u_1. The condition $u_3 = U_3$ as $x_2 \to \infty$ requires that:

$$u_3 = U_3 u_1/U_1 \tag{169}$$

In case the velocity U_3 is a function of x_1, then the pressure gradient in the x_3 direction is not zero.

$$-\rho^{-1}\partial_3 p = U_1 \partial_1 U_3 \tag{170}$$

The momentum equation for the x_3 direction 5̊ now becomes:

$$u_1 \partial_1 u_3 + u_2 \partial_2 u_3 = U_1 \partial_1 U_3 + \nu \partial_2^2 u_3 \tag{171}$$

The above equation is identical to the equation we had solved earlier 5̊ in the absence of a pressure gradient, except for the additonal inhomogeneous term. The general solution is the same as 8̊, but there is an additional particular solution. Since the velocity U_3 is proportional to x_1, we can try a solution of the form:

$$u_3 = U_{30} u_1 / U_1 + U_{31} x_1 g(\eta) \tag{172}$$

Inserting this into 1̊0, we get the following equation for $g(\eta)$:

$$g'' + (1/2) f g' - f' g - 1 = 0 \tag{173}$$

This equation can be solved for the unknown function $g(\eta)$, since the function $f(\eta)$ is known. The boundary conditions are $g = 0$ at $\eta = 0$, and $g = 0$ at $\eta \to \infty$.

2. The equations for the two components of the velocity are,

$$u_x = U f'(\eta)$$

$$u_y = \frac{1}{2}\left(\frac{\nu U}{x}\right)^{1/2} (\eta f'(\eta) - f(\eta))$$

where $\eta = (y/(\nu x/U)^{1/2})$. We use a similarity form for the equations, $(T - T_0)/(T_\infty - T_0) = h(\eta)$. Inserting this into the temperature equation, we get,

$$\frac{d^2 h}{d\eta^2} + \mathrm{Pr} f(\eta) \frac{dh}{d\eta} = 0$$

This is the equation that has to be solved for the similarity solution $h(\eta)$.

If the Prandtl number is large, we need to rescale η in the governing equations. Note that the thermal boundary layer is small compared to the momentum boundary layer in this case, and so we are considering the limit $\eta \ll 1$. In this case, the leading approximation for $f(\eta)$ is $f(\eta) = f''(0)\eta^2$, since both $f(\eta)$ and $f'(\eta)$ are zero at $\eta = 0$. We assume a form of the dimensionless variable $\eta = \mathrm{Pr}^a\xi$, where ξ is the new scaled co-ordinate, and a is a number that will be determined by a balance between convection and diffusion. The governing equation becomes,

$$\mathrm{Pr}^{-2a}\frac{d^2h}{d\xi^2} + Pr^{1+a}f''(0)\xi^2\frac{dh}{d\xi} = 0$$

It is clear from the above that $a = -1/3$, and the dimensionless variable ξ has to be defined as $\xi = \mathrm{Pr}^{-1/3}\eta$. The governing equation for the scaled temperature field now becomes,

$$\frac{d^2h}{d\xi^2} + f''(0)\xi^2\frac{dh}{d\xi} = 0$$

This equation can be easily solved to obtain,

$$h = \frac{1 - \int_0^\xi d\xi \exp\left(-f''(0)\xi^3/3\right)}{1 - \int_0^\infty d\xi \exp\left(-f''(0)\xi^3/3\right)}$$

The flux at the surface is can be determined as,

$$\begin{aligned} q &= -k\frac{dT}{dy}\bigg|_{y=0} \\ &= \mathrm{Re}^{1/2}\mathrm{Pr}^{1/3}\frac{k(T_0 - T_\infty)}{L}\frac{dh}{d\xi}\bigg|_{\xi=0} \end{aligned} \tag{174}$$

Thus, the Nusselt number is proportional to $\mathrm{Re}^{1/2}\mathrm{Pr}^{1/3}$ in the limit of high Prandtl number.

3. The velocity scale, determined from balancing the inertial and buoyancy terms, is $(\beta(T_1 - T_0)gh/\rho)^{1/2}$, where the temperature is scaled by $(T_1 - T_0)$. The pressure scale is $(\beta(T_1 - T_0)gh)$. Scaled this way, the momentum conservation equation becomes,

$$\frac{\partial \mathbf{u}^*}{\partial t^*} + \mathbf{u}^*.\nabla\mathbf{u}^* = -\nabla^* p^* + \frac{\nu}{h\sqrt{\beta g\rho(T_1 - T_0)}}\nabla^{*2}\mathbf{u}^*$$

In terms of the Gashof number $(\nu^2/h^3\beta\rho g(T_1 - T_0))$, this equation can be expressed as,

$$\frac{\partial \mathbf{u}^*}{\partial t^*} + \mathbf{u}^*.\nabla\mathbf{u}^* = -\nabla^* p^* + \mathrm{Gr}^{-1/2}\nabla^{*2}\mathbf{u}^*$$

The temperature equation is,

$$\frac{\partial T^*}{\partial t} + \mathbf{u}.\nabla T^* = \alpha\nabla^2 T^*$$

Introducing the scaled velocity and length, we obtain,

$$\frac{\partial T^*}{\partial t^*} + \mathbf{u}^*.\nabla^* T^* = \frac{\alpha}{h\sqrt{h\beta g(T_1 - T_0)/\rho}}\nabla^2 T^*$$

This can be expressed in terms of the Grashof number and Prandtl number as,

$$\frac{\partial T^*}{\partial t^*} + \mathbf{u}^*.\nabla^* T^* = \frac{1}{\mathrm{Pr}\mathrm{Gr}^{1/2}}\nabla^2 T^*$$

4. The boundary layer equations are,

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0$$

$$u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} + u_z\frac{\partial u_x}{\partial z} = -\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u_x}{\partial y^2}$$

$$\frac{\partial p}{\partial y} = 0$$

$$u_x\frac{\partial u_z}{\partial x} + u_y\frac{\partial u_z}{\partial y} + u_z\frac{\partial u_z}{\partial z} = -\frac{\partial p}{\partial z} + \nu\frac{\partial^2 u_z}{\partial y^2}$$

For the outer flow, the pressure gradients are given by,

$$\frac{\partial p}{\partial x} = -A^2 x$$

$$\frac{\partial p}{\partial z} = -B^2 z$$

From the mass conservation equation, we have,

$$\frac{\partial u_y}{\partial y} = -Af'(\eta) - Bg'(\eta)$$

This can be expressed in terms of the similarity variable to obtain

$$\sqrt{\frac{A}{\nu}} \frac{\partial u_y}{\partial \eta} = -Af'(\eta) - Bg'(\eta)$$

This can be integrated once, with the boundary condition $u_y = 0$ at $\eta = 0$ to obtain,

$$u_y = -\sqrt{\nu/A}(Af(\eta) + Bg(\eta)$$

The above expressions are substituted into the x momentum conservation equation to obtain,

$$A^2 x f^{'2} - \sqrt{\nu/A}(Af + Bg)(Ax/\sqrt{\nu/A})f'' = A^2 x + \nu(Ax/(\nu/A))f'''$$

Dividing throughout by $A^2 x$, we obtain,

$$f''' + f''(f + (B/A)g) + (1 - f^{'2}) = 0$$

The above expressions can be substituted into the g momentum conservation equations to obtain,

$$B^2 z g^{'2} - \sqrt{\nu/A}(Af + Bg)(Bz/\sqrt{\nu/A})g'' = B^2 z + \nu(Bz/(\nu/A))g'''$$

Dividing throughout by $B^2 z$, we obtain,

$$g''' + g''(f + (B/A)g) + (B/A)(1 - g^{'2}) = 0$$

Thus, a similarity solution can be obtained that depends only upon the ratio (A/B).