ACVGF

Then, one could write the general linear model in (9) as

\[ v[k] = \sum_{n=-\infty}^{\infty} h[n]q^{-n}e[k] = H(q^{-1}) \]  

(15)

where

\[ H(q^{-1}) = \sum_{n=-\infty}^{\infty} h[n]q^{-n} \]  

(16)

is known as the transfer function operator
Auto-covariance generating function

The auto-covariance generating function is defined as

\[ g(\sigma(z)) = \sum_{l=-\infty}^{\infty} \sigma_{vu}[l]z^{-l} \]  

(17)

where \( z \) is a variable.
ACVGF . . . contd.

The key use of this ACVF generating function stems from the fact that it can be computed directly from the MA representation of the random process.

\[ v[k] = H(q^{-1})e[k] \]  (18)

\[ g_{\sigma}(z) = \sigma^2_{\epsilon} H(z^{-1})H(z) \]  (19)

where \( H(z^{-1}) \) is obtained by replacing the operator \( q^{-1} \) in \( H(q^{-1}) \) with the variable \( z^{-1} \)
Example: ACVF of an MA(2) process

**Problem:** Compute the ACVF of an MA(2) process

\[ v[k] = e[k] + h_1 e[k - 2] + h_2 e[k - 2] \]

**Solution:** First observe that

\[ H(q^{-1}) = 1 + h_1 q^{-1} + h_2 q^{-2} \]

To compute the ACVF, construct the ACVGF by computing the product

\[ g_\sigma(z) = \sigma_e^2 H(z^{-1})H(z) = \sigma_e^2 (1 + h_1 z^{-1} + h_2 z^{-2})(1 + h_1 z + h_2 z^2) \]

\[ = \sigma_e^2 (h_2 z^{-2} + (h_1 + h_1 h_2) z^{-1} + (1 + h_1^2 + h_2^2) + (h_1 + h_1 h_2) z + h_2 z^2) \]
ACVGF of an MA(2) process

Comparing with equation (17) and reading off the coefficients of $z^{-l}$, we obtain

$$\sigma_{vv}[l] = \begin{cases} 
(1 + h_1^2 + h_2^2)\sigma_e^2, & l = 0 \\
(h_1 + h_1 h_2)\sigma_e^2, & l = 1 \\
h_2\sigma_e^2, & l = 2 \\
0, & |l| \geq 3 
\end{cases}$$

Thus, as expected, the ACVF of an MA(2) process vanishes at all lags $|l| > 2$
Auto-Regressive (AR) processes: ACF

The second class of processes that we consider are the **auto-regressive (AR) processes**.

For illustration, consider a first-order, i.e., AR(1) process:

\[ v[k] = -d_1 v[k - 1] + e[k] \tag{21} \]

where \( e[k] \) is the zero-mean GWN process of variance \( \sigma^2_e \) and \( d_1 \) is a finite constant.

- The current state is a linear function of the past state plus the unpredictable \( e[k] \).
- Assume \( |d_1| < 1 \) (a condition required for stationarity of \( v[k] \)).
The theoretical ACF can be now obtained using the definition in (2)
Observe that $\mu_e = 0 \implies \mu_v = 0$.

$$
\sigma_{vv}[l] = E(v[k]v[k - l]) \\
= -d_1 E(v[k - 1]v[k - l]) + E(e[k]v[k - l]) \\
= \phi_1 \sigma_{vv}[l - 1] + \sigma_{ev}[l]
$$

where $\sigma_{ev}[l]$ is the cross-covariance function, i.e., the covariance between $e[k]$ and $v[k - l]$ (see the definition of CCVF shortly)
ACF of an AR(1) process ... contd.

- By symmetry property of $\sigma_{uv}[l]$, it is sufficient to work out the derivation for $l \geq 0$. To complete the derivation, we first evaluate $\sigma_{ev}[l]$ for $l \geq 0$.

- A careful examination of (21) reveals that $v[k-l]$ contains effects of only past $e[k]$. By definition of WN, therefore, $\sigma_{ev}[l] = 0, l > 0$. 
ACF of an AR(1) process \ldots \text{contd.}

To obtain $\sigma_{ev}[0]$, multiply both sides of (21) with $e[k]$ and take expectations on both sides to yield,

\[ E(e[k]v[k]) = -d_1E(e[k]v[k-1]) + E(e[k]e[k]) \]
\[ = \sigma_e^2 \]

using the same arguments as above. Thus, we have the following set of equations

\[
\begin{align*}
\sigma_{vv}[0] &= -d_1\sigma_{vv}[-1] + \sigma_{ev}[0] \\
&= -d_1\sigma_{vv}[1] + \sigma_e^2 \\
\sigma_{vv}[1] &= -d_1\sigma_{vv}[0]
\end{align*}
\]
ACF of an AR(1) process . . . contd.

Solving equations for $\sigma_{vv}[0]$ and $\sigma_{vv}[1]$ simultaneously gives

\[
\begin{align*}
\sigma_{vv}[0] &= \frac{\sigma_e^2}{1 - d_1^2} \\
\rho_{vv}[l] &= (-d_1)^{|l|} \quad \forall \quad |l| \geq 1
\end{align*}
\] (22)
ACF of an AR(1) process

- Shown adjacent is the plot of the ACF of an AR(1) process with $d_1 = -0.5$
- In general whenever $|d_1| < 1$, we have that

The ACF of an AR(1) process exhibits exponential decay
The ACF measures linear dependencies between observations of a time-series.

For a stationary process, the ACF is a symmetric function.

The ACF coefficients at any lag determine the optimal linear model for $x[k]$ in terms of its past.

For an MA($M$) process, the ACF abruptly vanishes after lags $|l| > M$.

For an AR($P$) process, the ACF dies down only exponentially.

The ACF satisfies the same difference equation as the random process itself.