1. Consider a continuously operated fermenter described by the following equations

\[ \frac{dX}{dt} = -DX + \mu X \] 
\[ \frac{dS}{dt} = D(S_f - S) - \frac{1}{Y_{X/S}} \mu X \] 
\[ \frac{dP}{dt} = -DP + (\alpha \mu + \beta)X \]

where \( X \): effluent cell-mass or biomass concentration, \( S \): substrate concentration, \( P \): product concentration, \( D \): dilution rate, \( S_f \): feed substrate concentration, \( Y_{X/S} \): cell-mass yield, \( \alpha \) and \( \beta \):yield parameters for the product, \( \mu_m \): maximum specific growth rate, \( P_m \): product saturation constant, \( K_m \): substrate saturation constant, \( K_i \): substrate inhibition constant.

(a) The vectors of state (dependent) variable (\( x \)) and input (independent) variables (\( u \)) for this problem are defined as (2 mark)

(a) \( x = \begin{bmatrix} X & S_f & P \end{bmatrix}^T \); \( u = \begin{bmatrix} D & S \end{bmatrix}^T \)

(b) \( x = \begin{bmatrix} X & S & P \end{bmatrix}^T \); \( u = \begin{bmatrix} D & S_f \end{bmatrix}^T \)

(c) \( x = \begin{bmatrix} \mu & S & P \end{bmatrix}^T \); \( u = \begin{bmatrix} X & S_f \end{bmatrix}^T \)

(d) \( x = \begin{bmatrix} \mu & S & P \end{bmatrix}^T \); \( u = \begin{bmatrix} D & S_f \end{bmatrix}^T \)

(b) Consider a situation where \( D \) and \( S_f \) are fixed at \( D = 0.16 \) and \( S_f = 23 \) and values of \( (Y_{X/S}, \alpha, \beta, \mu_m, P_m, K_m, K_i) \) are given. It is desired to estimate \( X, S \) and \( P \) when all the transients have vanished and the system has come to a steady state. The resulting set of equations can be classified as (1.5 marks)

(a) ODE-Initial Value Problem (b) ODE-Boundary Value Problem (c) Nonlinear Algebraic Equations (d) Differential Algebraic Equations (e) Linear Algebraic Equations
(c) Consider a situation where at $t = 0$ we have $X = 7.0$, $S = 25$, $P = 24$ and values of $(Y_{X/S}, \alpha, \beta, \mu_m, P_m, K_m, K_i)$ are given. From $t = 0$ to 100, we change $D(t)$ as $D(t) = 0.16 + 0.02sin(0.2t)$ while holding $S_f = 23$. It is desired to find time variation of $X(t), S(t)$ and $P(t)$ as a function of time over $[0, 100]$. The resulting set of equations can be classified as (1.5 marks) (a) ODE-Initial Value Problem (b) ODE-Boundary Value Problem (c) Nonlinear Algebraic Equations (d) Differential Algebraic Equations (e) Linear Algebraic Equations

2. Let $X$ represent set of continuous functions on interval $0 \leq t \leq 1$ with inner product defined as

$$\langle x(t), y(t) \rangle = \int_0^1 t(1-t)x(t)y(t)dt$$

Given a set of four linearly independent vectors

$$x^{(1)}(t) = 1; \quad x^{(2)}(t) = t$$

find orthonormal set of vectors $e^{(1)}(t)$ and $e^{(2)}(t)$. (5 marks)

3. Show that in $C[-1,1]$ the following function

$$\langle f(t), g(t) \rangle = \max_t |x(t)y(t)|$$

cannot define an inner product. (5 marks)