1. Tube side output temperature measurements, \( (T_1, T_2, \ldots, T_n) \), have been collected from a shell and tube heat exchanger system operating at a steady state operating point. It is desired to fit a model of the form

\[
T_i = \beta + e_i
\]

using this data where constant \( \beta \) is an unknown parameter and \( e_i \) represents the approximation error.

(a) Show that the least square estimate of parameter \( \beta \) can be obtained by projecting vector \( \mathbf{U} = \begin{bmatrix} T_1 & T_2 & \cdots & T_n \end{bmatrix}^T \)
on vector \( \mathbf{a} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T \)

Also, show that the least square estimate of \( \beta \) correspond to the mean of the temperature data. (2 marks)

(b) The horizontal line \( T = 3 \) is closest to vector \( \mathbf{U} = \begin{bmatrix} 1 & 2 & 6 \end{bmatrix}^T \). Check that the vector

\[
\mathbf{p} = \begin{bmatrix} 3 & 3 & 3 \end{bmatrix}^T
\]
is perpendicular to error vector, \( \mathbf{e} = \mathbf{U} - \mathbf{p} \), and find the projection matrix \( \mathbf{P}_r = A (A^T A)^{-1} A^T \), which generates the projection vector \( \mathbf{p} \) from vector \( \mathbf{U} \). Also, state what is rank of the projection matrix \( \mathbf{P}_r \). (3 marks)

2. Relative height of a terrain (in meters) is measured at four corners \((1,0), (0,1), (-1,0)\) and \((0,-1)\) of a square field and was found to be 0, 1, 3, and 4 meters, respectively. Find the plane that gives best fit to four points in the least square sense. The equation for the plane is given as follows

\[
h = \alpha + \beta x + \delta y
\]

In other words, find the least square estimates of parameters \((\alpha, \beta, \delta)\) using the given data. Also, show that, at the center of the square, we have \( \hat{h} = \bar{\alpha} + \bar{\beta} x + \bar{\delta} y = \text{mean of heights at the four corners.} \) (6 marks)

3. Consider the following coupled differential equations characterize a system

\[
\begin{align*}
\frac{d^2 u}{dz^2} - u \frac{dv}{dz} & = 5u \sin(v) \quad \text{for} \quad 0 < z < 1 \\
\frac{d^2 v}{dz^2} + v \frac{du}{dz} & = uv^2 \quad \text{for} \quad 0 < z < 1 
\end{align*}
\]

together with following boundary conditions

\[
\begin{align*}
z = 0 : u(0) = 0 \quad \text{and} \quad v(0) = 1; \\
z = 1 : du(1)/dz = 0 \quad \text{and} \quad dv(1)/dz = 2(v(1) - 10)
\end{align*}
\]
Obtain a set of nonlinear algebraic equations using orthogonal collocation with 2 collocation points and arrange them in generic form $F(\mathbf{x}) = \mathbf{0}$. What is vector $\mathbf{x}$ in this case? (6 marks)

Note: The collocation points are at $z_1 = 0.21$ and $z_2 = 0.79$ and the corresponding $S$ and $T$ matrices are as follows

$$S = \begin{bmatrix}
-7 & 8.2 & -2.2 & 1 \\
-2.7 & 1.7 & 1.7 & -0.7 \\
0.7 & -1.7 & -1.7 & 2.7 \\
-1 & 2.2 & -8.2 & 7
\end{bmatrix}; \quad T = \begin{bmatrix}
24 & -37.2 & 25.2 & -12 \\
16.4 & -24 & 12 & -4.4 \\
-4.4 & 12 & -24 & 16.4 \\
-12 & 25.2 & -37.2 & 24
\end{bmatrix}$$

4. Find an approximate solution of ODE-BVP

$$\frac{d^2 u}{dz^2} + zu = 1$$

$B.C.: u(0) = u(1) = 0$

using Gelarkin’s method and following form of the approximate solution

$$\hat{u}(z) = az(1-z) + bz^2(1-z)$$

(8 marks)

**Note:** (a) Inner product is defined as

$$\langle f(z), g(z) \rangle = \int_0^1 f(z)g(z)dz$$

(b) Gelarkin’s Method: Given approximate solution as linear combination of basis functions

$$\{\hat{u}^{(1)}(z), \hat{u}^{(2)}(z), \ldots, \hat{u}^{(m)}(z)\}$$

solve for the following set of $m$ algebraic equations simultaneously

$$\langle \hat{u}^{(i)}(z), L\hat{u}(z) - f(z) \rangle = 0 \quad \text{for } i = 1, 2, \ldots, m$$