Consider a system governed by the following set of nonlinear ODEs

\[
\frac{dX_1}{dt} = 1.1226 \times 10^{-2} \left( \sqrt{X_3} - \sqrt{X_1} \right) + 8.325 \times 10^{-2} U_1 \quad (1)
\]

\[
\frac{dX_2}{dt} = 7.8859 \times 10^{-3} \left( \sqrt{X_4} - \sqrt{X_2} \right) + 6.2812 \times 10^{-2} U_2 \quad (2)
\]

\[
\frac{dX_3}{dt} = -1.1226 \times 10^{-2} \sqrt{X_3} + 4.7857 \times 10^{-2} U_2 \quad (3)
\]

\[
\frac{dX_4}{dt} = -7.8859 \times 10^{-3} \sqrt{X_4} + 3.1219 \times 10^{-2} U_1 \quad (4)
\]

Initial state of the plant is

\[
X(0) = \begin{bmatrix} 11 & 13.5 & 2.4 & 3 \end{bmatrix}^T
\]

A discrete linear perturbation model for this system in the neighborhood of steady state operating point

\[
\bar{X} = \begin{bmatrix} 12.4 & 12.7 & 1.8 & 1.4 \end{bmatrix}^T
\]

\[
\bar{U} = \begin{bmatrix} 3 & 3 \end{bmatrix}^T
\]

is given as follows

\[
x(k + 1) = \begin{bmatrix} 0.9225 & 0 & 0.1874 & 0 \\
0 & 0.946 & 0 & 0.1492 \\
0 & 0 & 0.8046 & 0 \\
0 & 0 & 0 & 0.8465 \end{bmatrix} x(k) + \begin{bmatrix} 0.4003 & 0.0235 \\
0.0121 & 0.304 \\
0 & 0.214 \\
0.1439 & 0 \end{bmatrix} u(k)
\]

Measured outputs are related to the states as follows

\[
y(t) = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\
0 & 0.5 & 0 & 0 \end{bmatrix} x(t) + v(k)
\]

where \( v(k) \) are zero mean Gaussian white noise sequence with

\[
Cov[v(k)] = \begin{bmatrix} (0.05)^2 & 0 \\
0 & (0.06)^2 \end{bmatrix}
\]
• **Plant Simulation**: Simulate the plant in open loop for \( k = 1,2,\ldots,100 \) samples under the following input conditions

- Generate the *known component* of the manipulated input signal, \( u(k) \), as follows

\[
\begin{align*}
0 \leq k \leq 30 & \quad u(k) = \begin{bmatrix} 3 \\ 3 \end{bmatrix}^T \\
31 \leq k \leq 65 & \quad u(k) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}^T \\
66 \leq k \leq 100 & \quad u(k) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}^T
\end{align*}
\]

- Unmeasured disturbance: The manipulated input entering the plant dynamics is \([u(k) + w(k)]\) where \( w(k) \) is a zero mean Gaussian white noise sequences with

\[
Q = Cov [w(k)] = \begin{bmatrix} (0.06)^2 & 0 \\ 0 & (0.07)^2 \end{bmatrix}
\]

• While simulating the plant dynamics, give user choice to simulate the plant either as

\[
\begin{align*}
\mathbf{x}(k+1) &= \Phi \mathbf{x}(k) + \Gamma [u(k) + w(k)] \\
\mathbf{x}(0) &= \mathbf{X}(0) - \mathbf{X}
\end{align*}
\]

OR by solving nonlinear ODEs (1-4) with the given initial condition. In the later case, the input entering the plant is determined as follows

\[
U(k) = \overline{U} + u(k) + w(k) \quad \text{for} \ kT \leq t < (k + 1)T
\]

where \( T \) represents sampling interval. Use \( T = 5 \) units.

• **State Estimation** Implement Kalman predictor by assuming

\[
\begin{align*}
\hat{\mathbf{x}}(0|0) &= \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T \\
P(0|0) &= 10 \mathbf{I}
\end{align*}
\]

and estimate sequence \( \{\hat{\mathbf{x}}(k|k) : k = 1,\ldots,100\} \).

- Graphically compare true \( \mathbf{x}_i(k) \) v/s time and estimated states \( \hat{\mathbf{x}}_i(k|k) \) v/s time for \( i = 1,2 \) (i.e. the unmeasured states).

- Plot state estimation error \( \varepsilon_i(k|k) \) v/s time for \( i = 1,2 \).
Kalman filter

\[
P(k|k-1) = \Phi P(k-1|k-1) \Phi^T + \Gamma_d Q \Gamma_d^T
\]

\[
L_e(k) = P(k|k-1) C^T [C P(k|k-1) C^T + R]^{-1}
\]

\[
\hat{x}(k|k-1) = \Phi \hat{x}(k-1|k-1) + \Gamma u(k-1)
\]

\[
e(k) = y(k) - C \hat{x}(k|k-1)
\]

\[
\hat{x}(k|k) = \hat{x}(k|k-1) + L_e e(k)
\]

\[
P(k|k) = [I - L(k) C(k)] P(k|k-1)
\]

**Note:** Since the unmeasured disturbance is in the manipulated inputs, \( \Gamma_d = \Gamma \) in this example.