Consider a system governed by the following set of nonlinear ODEs

\[
\begin{align*}
\frac{dX_1}{dt} &= 1.1226 \times 10^{-2} \left( \sqrt{X_3} - \sqrt{X_1} \right) + 8.325 \times 10^{-2} U_1 \\
\frac{dX_2}{dt} &= 7.8859 \times 10^{-3} \left( \sqrt{X_4} - \sqrt{X_2} \right) + 6.2812 \times 10^{-2} U_2 \\
\frac{dX_3}{dt} &= -1.1226 \times 10^{-2} \sqrt{X_3} + 4.7857 \times 10^{-2} U_2 \\
\frac{dX_4}{dt} &= -7.8859 \times 10^{-3} \sqrt{X_4} + 3.1219 \times 10^{-2} U_1 
\end{align*}
\]  

Initially state of the plant is

\[ X(0) = \begin{bmatrix} 11 & 13.5 & 2.4 & 3 \end{bmatrix}^T \]

A discrete linear perturbation model for this system in the neighborhood of steady state operating point

\[ X = \begin{bmatrix} 12.4 & 12.7 & 1.8 & 1.4 \end{bmatrix}^T \text{ and } U = \begin{bmatrix} 3 & 3 \end{bmatrix}^T \]

is given as follows

\[
x(k + 1) = \begin{bmatrix} 0.9225 & 0 & 0.1874 & 0 \\ 0 & 0.946 & 0 & 0.1492 \\ 0 & 0 & 0.8046 & 0 \\ 0 & 0 & 0 & 0.8465 \end{bmatrix} x(k) + \begin{bmatrix} 0.4003 & 0.0235 \\ 0.0121 & 0.304 \\ 0 & 0.214 \\ 0.1439 & 0 \end{bmatrix} u(k)
\]

Measured outputs are related to the states as follows

\[ y(t) = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \end{bmatrix} x(t) + v(k) \]

where \( v(k) \) are zero mean Gaussian white noise sequence with

\[ Cov[v(k)] = \begin{bmatrix} (0.05)^2 & 0 \\ 0 & (0.06)^2 \end{bmatrix} \]

- **Plant Simulation:** Simulate the plant in open loop for \( k = 1, 2, \ldots, 100 \) samples under the following input conditions
– Generate the known component of the manipulated input signal, $u(k)$, as follows

$$
0 \leq k \leq 30 \quad u(k) = \begin{bmatrix} 3 & 3 \end{bmatrix}^T \\
31 \leq k \leq 65 \quad u(k) = \begin{bmatrix} 4 & 2 \end{bmatrix}^T \\
66 \leq k \leq 100 \quad u(k) = \begin{bmatrix} 2 & 4 \end{bmatrix}^T
$$

– Unmeasured disturbance: The manipulated input entering the plant dynamics is $[u(k) + w(k)]$ where $w(k)$ is a zero mean Gaussian white noise sequences with

$$
Q = \text{Cov}[w(k)] = \begin{bmatrix} (0.06)^2 & 0 \\
0 & (0.07)^2 \end{bmatrix}
$$

• While simulating the plant dynamics, give user choice to simulate the plant either as

$$
x(k+1) = \Phi x(k) + \Gamma [u(k) + w(k)] \\
x(0) = X(0)-X
$$

OR by solving nonlinear ODEs (1-4) with the given initial condition. In the later case, the input entering the plant is determined as follows

$$
U(k) = \bar{U} + u(k) + w(k) \quad \text{for } kT \leq t < (k+1)T
$$

where $T$ represents sampling interval. Use $T = 5$ units.

• **State Estimation** Implement Kalman predictor by assuming

$$
\hat{x}(0|1) = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T \\
P(0|1) = 10I
$$

and estimate sequence $\{\hat{x}(k|k-1) : k = 1, \ldots, 100\}$.

– Graphically compare true $x_i(k)$ v/s time and estimated states $\hat{x}_i(k|k-1)$ v/s time for $i = 1, 2$ (i.e. the unmeasured states).

– Plot estimation error $\varepsilon_i(k|k-1)$ v/s time for $i = 1, 2$.

**Kalman Predictor**

$$
L_p(k) = \Phi P(k|k-1)C^T [CP(k|k-1)C^T + R]^{-1} \\
e(k) = y(k) - C\hat{x}(k|k-1) \\
\hat{x}(k+1|k) = \Phi \hat{x}(k|k-1) + \Gamma_u u(k) + L_p e(k) \\
P(k+1|k) = \Phi P(k|k-1)\Phi^T + \Gamma_d Q \Gamma_d^T - L(k)CP(k|k-1)\Phi^T
$$

**Note:** Since the unmeasured disturbance is in the manipulated inputs, $\Gamma_d = \Gamma$ in this example.