1. Consider an ARMA process

\[ v(k) = \alpha v(k - 1) + e(k) + \beta e(k - 1) \]  \hspace{1cm} (1)

where \( \{e(k)\} \) is a zero mean white noise process with variance \( \lambda^2 \). It can be shown that stochastic process \( \{v(k)\} \) has zero mean.

(a) Derive expressions for cross-covariance \( r_{ve}(1) = E[v(k)e(k - 1)] \) \hspace{1cm} (2 marks)

(b) Derive expressions for auto-covariance \( r_v(1) = E[v(k)v(k - 1)] \). \hspace{1cm} (4 marks)

2. Consider Box-Jenkin’s model

\[ y(k) = q^{-1} + 0.5q^{-2} u(k) + \frac{1 + 0.5q^{-1}}{(1 - 0.8q^{-1})} e(k) \]

Derive one step prediction

\[ \hat{y}(k|k - 1) = [H(q)]^{-1} G(q) u(k) + [1 - (H(q))^{-1}] y(k) \]

\[ y(k) = \hat{y}(k|k - 1) + e(k) \]

and express dynamics of \( \hat{y}(k|k - 1) \) as a time domain difference equation. \hspace{1cm} (6 marks)

3. Consider a coupled tank system in which dynamics of levels in the two tanks is governed by

\[ \frac{dx}{dt} = \begin{bmatrix} -3 & 2 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u \]

where \( x \) denotes perturbations in level and \( u \) denotes perturbations in inlet flow.

(a) It is desired to control this system (at the setpoint equal to the origin) using a feedback control law of the form

\[ u = - \begin{bmatrix} \alpha & \beta \end{bmatrix} x \]

Determine the state space model (differential equation) that governs the closed loop dynamics in terms of unknowns \( \alpha, \beta \). \hspace{1cm} (2 marks)

(b) Determine, if it exists, controller gains \( \begin{bmatrix} \alpha & \beta \end{bmatrix} \) such that the state transition matrix for the closed loop system has eigenvalues at the roots of the following quadratic equation \hspace{1cm} (4 marks)

\[ \lambda^2 + 11\lambda + 30 = 0 \]
4. Consider a **continuous time** linear perturbation model

\[ \frac{dx}{dt} = Ax + Bu \]

\[
A = \begin{bmatrix}
-2 & 1/2 & 1/2 \\
1 & -3/2 & -1/2 \\
1 & 1/2 & -5/2
\end{bmatrix}
\quad \text{and} \quad
B = \begin{bmatrix}
-1 & 1 \\
0 & 1 \\
2 & -1
\end{bmatrix}
\]

Eigenvalues of matrix \( A \) are -1, -2 and -3 and the continuous time system is asymptotically stable. Suppose discretization of the continuous time system is carried out using the Euler’s method i.e. \( [\Phi]_{Euler} = I + TA \). Then, find the range of sampling time \( T \) for which the discrete time model will retain the stability characteristics of the continuous time system. (7 marks)

**Hint**: If matrix \( A \) is diagonalizable, can you relate eigenvalues of \( A \) with eigenvalues of \( [\Phi]_{Euler} \)?