Optimal Control, Guidance and Estimation

Lecture – 26

Linear Quadratic Observer
An Overview of State Estimation

Prof. Radhakant Padhi
Dept. of Aerospace Engineering
Indian Institute of Science - Bangalore

Topics

- Motivation for State Estimation

- Linear-Quadratic (LQ) Observer Design

- State Estimation using Kalman Filter: An Overview
Why State Estimation?

- State feedback control designs need the state information for control computation.
- In practice all state variables are not available for feedback. Possible reasons are:
  - Non-availability of sensors
  - Expensive sensors
  - Quality of sensor input is not acceptable due to noise (it’s an issue in output feedback control design as well)
- A state observer estimates the state variables based on the measurement of some of the output variables as well as the process information.
Other Application of Estimation

- Model development (parameter estimation, state-parameter estimation)

- Fault Detection and Identification (FDI)

- Estimation of states of other related systems to collect exogenous information (e.g. Target state estimation for missile guidance)

LQ Observer Design

Prof. Radhakant Padhi
Dept. of Aerospace Engineering
Indian Institute of Science - Bangalore
Observer

- An observer is a dynamic system whose output is an estimate of the state vector $X$
  - Full-order Observer
  - Reduced-order Observer

- **Observability** condition must be satisfied for designing an observer (this is true for filter design as well)

**Observer Design for Linear Systems**

Plant: $\dot{X} = AX + BU$

$Y = CX$ (sensor output vector)

Let the observed state be $\hat{X}$ and the Observer dynamics be:

$\dot{\hat{X}} = \hat{A}\hat{X} + \hat{B}U + K_Y$

Error: $\tilde{X} \triangleq (X - \hat{X})$

Observer Design for Linear Systems

Error Dynamics: \[
\hat{X} = \hat{X} - \hat{X} = (AX + BU) - (\hat{A}\hat{X} + \hat{B}U + K_c Y)
\]

Add and Substract \(\hat{AX}\) and substitute \(Y = CX\)
\[
\hat{X} = AX - \hat{AX} + \hat{AX} + BU - \hat{B}U - K_c CX
\]
\[
= (A - \hat{A})X + \hat{A}(X - \hat{X}) + (B - \hat{B})U - K_c CX
\]
\[
= \hat{A}\hat{X} + (A - \hat{A} - K_c C)X + (B - \hat{B})U
\]

Goals:
1. Make the error dynamics independent of \(X\)
   \(\because\) \(X\) can be large, even though \(\hat{X}\) may be small
2. Eliminate the effect of \(U\) from error dynamics

Observer Design for Linear Systems

This can be done by enforcing \(A - \hat{A} - K_c C = 0\) and \(B - \hat{B} = 0\)

Necessary and sufficient condition for the existence of \(K_c\):

The system should be “observable”.

This results in:
\[
\hat{A} = A - K_c C
\]
\[
\hat{B} = B
\]

Observer dynamics:
\[
\dot{\hat{X}} = A\hat{X} + BU + K_c (Y - C\hat{X})
\]
**Observer Design: Full Order**

- Order of the observer is same as that of the system in a full-order observer; i.e. all states are estimated, irrespective of whether they are measured or not.

- Goal: To obtain gain $K_e$ such that the error dynamics is asymptotically stable with sufficient speed of response.

**Comparison of Control and Observer Design Philosophies**

<table>
<thead>
<tr>
<th>Control Design</th>
<th>Observer Design</th>
</tr>
</thead>
<tbody>
<tr>
<td>CL Dynamics</td>
<td>CL Error Dynamics</td>
</tr>
<tr>
<td>$\dot{X} = (A - BK)X$</td>
<td>$\dot{X} = \hat{A}\hat{X} = (A - K_eC)\hat{X}$</td>
</tr>
<tr>
<td>Objective</td>
<td>Objective</td>
</tr>
<tr>
<td>$X(t) \to 0$, as $t \to \infty$</td>
<td>$\hat{X}(t) \to 0$, as $t \to \infty$</td>
</tr>
</tbody>
</table>

Notice that

$$\lambda(A - K_eC) = \lambda[(A - K_eC)^T]$$

$$= \lambda(A^T - C^TK_e^T)$$
**Algebraic Riccati Equation (ARE) Based Observer Design**

<table>
<thead>
<tr>
<th>System</th>
<th>Dual System</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \dot{X} = AX + BU )</td>
<td>( \dot{Z} = A^T Z + C^T V )</td>
</tr>
<tr>
<td>( Y = CX )</td>
<td>( n = B^T Z )</td>
</tr>
<tr>
<td>( M = [B \mid AB \mid \cdots \mid A^{n-1} B] )</td>
<td>( M = [C^T \mid A^T C^T \mid \cdots \mid A^{T+1} C^T] )</td>
</tr>
<tr>
<td>( N = [C^T \mid A^T C^T \mid \cdots \mid A^{T+1} C^T] )</td>
<td>( N = [B \mid AB \mid \cdots \mid A^{n-1} B] )</td>
</tr>
</tbody>
</table>

**LQR Design**

\[ U = -KX \]

---

**ARE Based Observer Design**

**CL system (control design)**

\[
\dot{X} = (A - BK) X \\
X \to 0 \quad as \quad t \to \infty
\]

\[ K = R^{-1} B^T P, \quad P > 0 \]

where,

\[ PA + A^T P - PBR^{-1}B^T P + Q = 0 \]

**Error Dynamics**

\[
\dot{\bar{X}} = (A - K e C) \bar{X} \\
(A - K e C)^T = A^T - C^T K e^T
\]

**Analogous**

\[ K e^T = R^{-1} C P \]

where,

\[ PA^T + A^T C P + Q = 0 \]

**Observer Dynamics**

\[
\dot{\hat{X}} = A \hat{X} + BU + K e (Y - C \hat{X})
\]
Main Aspects of Estimation

- Prediction, Filtering and Smoothing: improves noisy measurements, typically due to process noise and measurement noise.
- Can also take care of limited inaccuracy or errors in system modeling (i.e. relying on estimated states leads to a robust controller)
- Parameter estimation (for system identification)
### Historical Development of Filtering Theory

- **Gauss (1795):** Least square estimation

- **Wiener filter (1940’s):**
  - Derived from variational calculus
  - Requires solution of integral equation
  - Difficult to use in practice

- **Kalman filter (1961):** It was a major breakthrough in linear estimation theory. The salient features include:
  - Recursive solution and suitable for online.
  - Many applications: Primarily developed for Apollo-mission. Later used in a variety of applications (e.g. flight control, submarines, GPS …etc.)

### Pioneers of Optimal Control

- **1700s**
  - Bernoulli
  - Euler *(Student of Bernoulli)*
  - Lagrange
  - **200 years later**....

- **1900s**
  - Pontryagin
  - Bellman
  - **Kalman**
About R. E. Kalman:

Education
Ref: Wikiepedia

- Born on 19 May 1930, in Budapest, Hungary (now is 80+).
- Emigrated to the United States in 1943
- Earned his bachelor's degree in 1953 and his master's degree in 1954, both from the Massachusetts Institute of Technology (MIT), in electrical engineering
- Earned his D.Sci. in 1957 at Columbia University in New York City

About R. E. Kalman
Ref: Wikiepedia

- Research Mathematician at the Research Institute for Advanced Studies in Baltimore, Maryland from 1958 until 1964… this is where his legendary paper appeared.
- Professor at Stanford University from 1964 until 1971
- Graduate Research Professor and the Director of the Center for Mathematical System Theory, at the University of Florida from 1971 until 1992
- Starting in 1973, he also held the chair of Mathematical System Theory at the Swiss Federal Institute of Technology (ETH) in Zürich, Switzerland.
- Currently, is professor emeritus at the Swiss Federal Institute of Technology in Zurich (ETHZ) as well as at the University of Florida (Gainesville), USA.
About R. E. Kalman
Ref: Wikiepedia

- Kálmán's ideas on filtering were initially met with vast skepticism, so much so that he was forced to do the first publication of his results in mechanical engineering (Transactions of the ASME~Journal of Basic Engineering, Vol. 82, 1960)

- Kálmán had more success in presenting his ideas, while visiting Stanley F. Schmidt at the NASA Ames Research Center in 1960. This led to the use of Kálmán filters during the Apollo program, and furthermore, in the NASA Space Shuttle, in Navy submarines, and in unmanned aerospace vehicles and weapons, such as cruise missiles.

......Rest is History!

Other Contributions of R. E. Kalman
Ref: http://www.cs.unc.edu/~welch/kalman/siam_sontag.html

- During the 1960s, Kalman was a leader in the development of a rigorous theory of control systems. Formulation and study of most fundamental state-space notions (including controllability, observability, minimality, realizability from input/output data, matrix Riccati equations, linear-quadratic control, separation principle etc.)

- During the 1970s Kalman played a major role in the introduction of algebraic and geometric techniques in the study of linear and nonlinear control systems.

- His work since the 1980s has focused on a system-theoretic approach to the foundations of statistics, econometric modeling, and identification, as a complement to his earlier studies of minimality and realizability.
About R. E. Kalman

- IEEE Medal of Honor in 1974 (IEEE’s highest honor)
- IEEE Centennial Medal in 1984
- Inamori foundation's Kyoto Prize in 1985 (Japanese Nobel prize!)
- Steele Prize of the American Mathematical Society in 1987
- Richard E. Bellman Control Heritage Award in 1997 (from American Automatic Control Council (AACC))
- National Academy of Engineering's Charles Stark Draper Prize on 19th Feb 2008 ($5,00,000 prize!).

Kalman Filtering....Applications

- Google search (20 Nov 2012)
  - “Kalman Filtering”: About 20,90,000 results!
  - “Prof. R. E. Kalman”: About 8,75,000 results!
- Application domains
  - Aerospace
  - Electrical
  - Chemical
  - Mechanical
  - Image processing
  - Robotics…. virtually every field
- A rich source: http://www.cs.unc.edu/~welch/kalman/
Kalman Filter:
Information Required & Task

- **Information Required**
  - System model (linear/linearized model)
  - Measurements and their statistical behaviors
  - Statistical models characterizing the process and measurement noise (typically zero-mean uncorrelated white noise)
  - Initial condition information for the states

- **Task:**
  - To Estimate (filter) the state by processing the measurement data and using the system model

---

Kalman Filter

System dynamics

\[
\dot{X} = AX + BU + GW \\
Y = CX + v
\]

\( w(t) \): Process noise that acts to disturb the plant
  (e.g. Wind gusts, unmodelled high-frequency dynamics )

\( v(t) \): Measurement noise (sensor noise)
Assumptions of Kalman Filter

\( w(t), v(t) : \) Zero-mean “white noise”

\( X(0) : \) Unknown

\( X(0) \sim (\hat{X}_0, P_0) \)

\( w(t) \sim (0, Q), \quad Q \geq 0 \)

\( v(t) \sim (0, R), \quad R > 0 \)

Kalman Filter

Estimator

\[ \dot{\hat{X}} = A\hat{X} + BU + K_e (Y - \hat{Y}) \]

Where

\[ \hat{Y} = E(CX + v) = 0 \]

\[ = E(CX) + E(v) = CE(X) = C\hat{X} \]
Kalman Filter

Error Covariance Matrix

\[ P(t) = E \left[ \tilde{X} \tilde{X}^T \right] \]

where

\[ \tilde{X} \triangleq X(t) - \hat{X}(t) \]

Note:

1. \( P(t) \) is a measure of uncertainty in the estimate

2. If the observer dynamics is asymptotically stable, and \( w(t), v(t) \)
   are stationary processes, the error will eventually reach a steady state

Key:

The gain \( K_e \) is chosen so that it minimizes the steady-state error covariance. The optimal gain will be a “constant matrix”

Kalman Filter (Mechanization)

Initialization

\[ \hat{X}(0) = \tilde{X}_0 \]

Kalman Gain

\[ K_e = PC^T R^{-1}, \quad P > 0 \]

Error Covariance ARE

\[ PA^T + AP - PC^T R^{-1} CP + GQG^T = 0 \]

Estimator (Filter) Dynamics

\[ \dot{\hat{X}} = A\hat{X} + BU + K_e(Y - C\hat{X}) \]
Linear Quadratic Gaussian (LQG) Control Design

**Philosophy:**
- Controller design: LQR method
- State Estimation: Kalman filter

\[
\text{Ideal: } U = -KX
\]
\[
\text{Usage: } \hat{U} = -K\hat{X}
\]

**Problem:**
Loss of robustness

**Remedy:**
Loop Transfer Recovery (LTR): LQG/LTR design

Extended Kalman Filtering (EKF):
An Overview

Prof. Radhakant Padhi
Dept. of Aerospace Engineering
Indian Institute of Science - Bangalore
Extended Kalman Filter (EKF)

- For nonlinear system models, Kalman Filter is not applicable
- How to predict the state vector and its error covariance?
- EKF is extension of Kalman filter via linearization
- It processes all available sensor measurements in estimating the value of states of interest using:
  * Knowledge of system and sensor dynamics.
  * Statistical models reflecting uncertainty in process noise and sensor noise
  * Some information regarding initial condition.

Nonlinear System Dynamics and EKF Design

System Dynamics
\[
\dot{X}(t) = f(X(t), U(t), t) + G(t)w(t) \quad E[w(t)w^T(\tau)] = Q(t)\delta(t-\tau)
\]

Output dynamics
\[
Y(t_i) = h(X(t_i), t_i) + v(t_i) \quad E[v_i v_j^T] = R \delta_{ij}
\]

It works in two step:

I. Time Update (‘Prediction’).
II. Measurement Update (‘Correction’).
Step I: Prediction from $t_{k-1}^-$ to $t_k^-$

- The optimal estimated states and $P$ are propagated, based on the previous values, the system dynamics, and the previous control inputs and errors of the actual system.

- Propagate the state equation (by numerical integration)

$$\dot{\hat{X}}(t) = f(\hat{X}(t), U(t), t)$$

- Propagate the error Covariance matrix

$$\dot{\hat{P}}(t) = P\Lambda + A^T P + Q$$ where, $A(t) = \left[ \frac{\partial f}{\partial \hat{X}} \right]$ $\hat{X}$

Step II: Filtering from $t_k^-$ to $t_k^+$

- Compute the filter gain

$$K_{\hat{X}} = P_{\hat{X}} C_{\hat{X}}^{-T} \left[ C_{\hat{X}}^{-T} P_{\hat{X}} C_{\hat{X}}^{-T} + R_k \right]^{-1}, \quad C_{\hat{X}}^{-} = \left[ \frac{\partial h}{\partial \hat{X}} \right]_{\hat{X}}$$

- Update the state vector and error covariance matrix

$$\hat{X}_k^+ = \hat{X}_k^- + K_{\hat{X}} \left[ Y_k - h(\hat{X}_k^-) \right]$$

$$P_k^+ = \left[ I - K_{\hat{X}} C_{\hat{X}}^{-} \right] P_k^-$$
Advantages/Limitations of EKF

- **Advantages:**
  - It works for a wide variety of practical problems
  - Its computationally very efficient

- **Limitations:**
  - Linearization can introduce significant error
  - No general convergence guarantee
  - Works in general; but in some cases its performance can be surprisingly bad
  - Unreliable for colour noise

- **Issues:**
  - Optimal measurement schedules
  - Parameter/Modeling uncertainties
  - Computational errors
  - Noise model (e.g. Non-Gaussian PDF)

Beyond EKF

- **Need**
  - Nonlinear systems
  - Non-Gaussian noises/inputs
  - Correlated noises

- **Characteristics of Such Filters**
  - Are often approximate
  - Sacrifices theoretical accuracy in favour of practical constraints and considerations like robustness, adaptation, numerical feasibility
  - Attempt to cover the limitations of EKF
Thanks for the Attention....!!