Lecture – 18

Time Response of Linear Dynamical Systems in State Space Form

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Solution of Linear Differential Equations

Linear systems:

Systems that obey the "Principle of superposition".

Uniqueness Theorem:

There is only one solution for linear systems.

Solution of Homogeneous Linear Differential Equation: Scalar case

System dynamics: $\dot{x} = ax, \quad x(t_0) = x_0$

Solution:

$$(dx/x) = a\,dt$$

$$\ln x = a\,t + \ln c$$

$$\ln(x/c) = a\,t$$

$$x = e^{at}c$$

Initial condition: $x_0 = e^{at_0}c, \quad c = e^{-at_0}x_0$

Hence, $$x(t) = e^{a(t-t_0)}x_0$$

Solution of Homogeneous Linear Differential Equation: Scalar case

Note:

$$e^{at} = 1 + at + \frac{a^2 t^2}{2!} + \frac{a^3 t^3}{3!} + \cdots$$

$$\text{If } \; t_0 = 0, \quad \text{then the solution is}$$

$$x(t) = e^{at} x_0$$

Solution of Homogeneous Linear Differential Equations

System dynamics: $\dot{X} = AX, \quad X(t_0) = X_0$

Guess solution: $X(t) = e^{At}C \;, \quad C = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix}^T$

$$e^{At} \triangleq I + At + A^2t^2/2! + A^3t^3/3! + \cdots$$

Verify (substitute the guess into the differential equation)

$$\left(\frac{d}{dt} e^{At} \right) C = A\left(e^{At} C \right)$$

Solution of Homogeneous Linear Differential Equations

A Result:

$$e^{At} = I + At + A^2t^2/2! + A^3t^3/3! + \cdots$$

$$\frac{d}{dt}\left(e^{At}\right) = 0 + A + A^2\left(2t/2!\right) + A^3\left(3t^2/3!\right) + \cdots$$

$$= A\left(I + At + A^2t^2/2! + A^3t^3/3! + \cdots\right)$$

$$= Ae^{At}$$

i.e. $\left(Ae^{At}\right)\ C = A\left(e^{At}C\right)$

Therefore $X(t) = e^{At}C$ is 'a' solution.

Hence, $X(t) = e^{At}C$ is 'the' solution.

Solution of Homogeneous Linear Differential Equations

- Applying the initial condition $X_0 = e^{At_0} C$

$$C = \left[e^{At_0} \right]^{-1} X_0$$

- Another result: $e^{A(t_1+t_2)} = e^{At_1} e^{At_2}$

(easy to show from definition)

Taking $t_1 = t_0$ and $t_2 = -t_0$, $I = e^{At_0} e^{-At_0}$

Thus $\left[e^{At_0} \right]^{-1} = e^{-At_0}$

Finally $$X(t) = e^{At} e^{-At_0} X_0 = e^{A(t-t_0)} X_0$$

Solution of Non-homogeneous Linear Differential Equations

Non-homogeneous system: $\dot{X} = AX + BU \; , \; X(t_0) = X_0$

Solution contains two parts:

- Homogeneous solution
- Particular solution

Homogeneous solution: $X_h(t) = e^{A(t-t_0)} X_0$

Particular solution: $X_p(t) = e^{At} C(t)$

Solution of Non-homogeneous Linear Differential Equations

$$\dot{X}_p = e^{At}\dot{C} + \cancel{Ae^{At}C} = \cancel{Ae^{At}C} + BU$$

$$\dot{C} = e^{-At}BU$$

$$C(t) = \int_{t_1}^{t} e^{-A\tau}BU(\tau)\,d\tau$$

$$X_p(t) = e^{At}C(t) = e^{At}\int_{t_1}^{t} e^{-A\tau}BU(\tau)\,d\tau$$

$$= \int_{t_1}^{t} e^{A(t-\tau)}BU(\tau)d\tau$$

Solution of Non-homogeneous Linear Differential Equations

Complete solution: $X(t) = X_h(t) + X_p(t)$

$$= e^{A(t-t_0)} X_0 + \int_{t_1}^{t} e^{A(t-\tau)} BU(\tau) d\tau$$

Initial condition: At $t = t_0$

$$X_0 = X_0 + \int_{t_1}^{t_0} e^{A(t-\tau)} BU(\tau)\, d\tau$$

$$\int_{t_1}^{t_0} e^{A(t-\tau)} BU(\tau)\, d\tau = 0$$

This suggests that $t_1 = t_0$

Solution of Non-homogeneous Linear Differential Equations

Complete solution: $$X(t) = e^{A(t-t_0)} X_0 + \int_{t_0}^{t} e^{A(t-\tau)} BU(\tau) d\tau$$

The integral term in the forced system solution is a *convolution integral.*

Note: If U is in feedback form $(U = -K X)$

$$\dot{X} = (A - BK) X = A_{CL} X$$

$$X(t) = e^{A_{CL}(t-t_0)} X_0$$

Solution of Non-homogeneous Linear Differential Equations: Some Comments

The solution results do not demand that $t \geq t_0$.
They are equally valid even if $t \leq t_0$.

The integral term in the forced system solution is a "convolution integral".
i.e. The contribution of input $U(t)$ is the convolution of $U(t)$ with $e^{At}B$.
Hence, the function $e^{At}B$ has the role of "impulse response" of the system whose output is $X(t)$ and input is $U(t)$.

The solution for output $Y(t)$ is also readily available from $X(t)$ and $U(t)$:

$$Y(t) = CX(t) + DU(t)$$

Example: Motion of a car without friction

The equation of motion is

$$m\ddot{x} = f(t)$$

$$\ddot{x} = (1/m)\, f(t) \qquad \text{Assumption: } m \text{ is constant.}$$

$$v \triangleq \dot{x}$$

$$\dot{X} = \begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x \\ v \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1/m \end{bmatrix}}_{B} f(t), \quad X(0) = \begin{bmatrix} x_0 \\ v_0 \end{bmatrix}$$

Example: Motion of a car without friction

$$X(t) = e^{At} X_0 + \int_0^t e^{A(t-\tau)} B\ f(\tau)\, d\tau$$

$$e^{At} = I + At + A^2 \frac{t^2}{2!} + \cdots = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$$e^{A(t-\tau)} = \begin{bmatrix} 1 & t-\tau \\ 0 & 1 \end{bmatrix}$$

$$e^{A(t-\tau)} B = \begin{bmatrix} 1 & t-\tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1/m \end{bmatrix} = \frac{1}{m} \begin{bmatrix} t-\tau \\ 1 \end{bmatrix}$$

Example: Motion of a car without friction

$$\begin{bmatrix} x(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} x_0 + v_0 t + \dfrac{1}{m}\displaystyle\int_0^t (t-\tau) f(\tau) d\tau \\ v_0 + \dfrac{1}{m}\displaystyle\int_0^t f(\tau) d\tau \end{bmatrix} = \begin{bmatrix} x_0 + v(t)t - \dfrac{1}{m}\displaystyle\int_0^t \tau \ f(\tau) d\tau \\ v_0 + \dfrac{1}{m}\displaystyle\int_0^t f(\tau) d\tau \end{bmatrix}$$

Special case: $f(t)/m = a$ (constant) and $\begin{bmatrix} x_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} x(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} x_0 + (v_0 + at)t - \dfrac{t^2}{2} a \\ v_0 + at \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2} a t^2 \\ at \end{bmatrix}$$

Evaluation of e^{At} : A Useful Result

Problem: $\dot{X} = AX, \qquad X(0) = X_0$

Solution using Laplace transform:

$$sX(s) - X_0 = AX(s)$$

$$\left(sI - A\right)X(s) = X_0$$

$$X(s) = \left(sI - A\right)^{-1} X_0$$

$$X(t) = L^{-1}\left[\left(sI - A\right)^{-1}\right] X_0$$

Solution known:

$$X(t) = e^{At} X_0$$

Comparing the two solutions:

$$e^{At} = L^{-1}\left[\left(sI - A\right)^{-1}\right]$$

Evaluation of e^{At} : How to compute it symbolically?

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad (sI - A) = \begin{bmatrix} s - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & s - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & s - a_{nn} \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{adj(sI - A)}{|sI - A|}$$

$$|sI - A| = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n$$

$$adj(sI - A) = E_1 s^{n-1} + E_2 s^{n-2} + \cdots + E_n$$

Symbolic computation of e^{At}

$$(sI - A)(sI - A)^{-1} = \frac{(sI - A)\left(E_1 s^{n-1} + E_2 s^{n-2} + \cdots + E_n\right)}{s^n + a_1 s^{n-1} + a_2 s^{n-1} + \cdots + a_n}$$

$$I\left(s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n\right) = (sI - A)\left(E_1 s^{n-1} + E_2 s^{n-2} + \cdots + E_n\right)$$

$$s^n I + a_1 s^{n-1} I + a_2 s^{n-2} I + \cdots + a_n I = s^n E_1 + s^{n-1}\left(E_2 - AE_1\right) + \cdots + s\left(E_n - AE_{n-1}\right) - AE_n$$

Equate the coefficients on both sides…

Symbolic computation of e^{At}

$$E_1 = I$$

$$E_2 - AE_1 = a_1 I$$

$$E_3 - AE_2 = a_2 I$$

$$\vdots$$

$$E_n - AE_{n-1} = a_{n-1} I$$

$$-AE_n = a_n I$$

This suggests a recursive algorithm!
(provided $a_1, \ldots, a_n$ are known)

$$\Downarrow$$

$$a_i = -\left(\frac{1}{i}\right) Tr\left(AE_i\right)$$

for $i = 1, \ldots, n$

Reference: T. Kailath, Linear Systems, Prentice-Hall, 1980.

Solution of Linear Time Varying Systems

Homogeneous Linear System $\dot{X} = A(t)X$

Solution: $X(t) = \varphi(t,\tau)\,X(\tau)$

$\varphi(t,\tau)$: State Transition Matrix (STM)

PROPERTIES OF STM

1. It satisfies linear differential equation $\dfrac{\partial \varphi}{\partial t} = A(t)\varphi(t,\tau)$

2. $\varphi(t,t) = I$

3. For any three time instants $\varphi(t_3,t_1) = \varphi(t_3,t_2)\,\varphi(t_2,t_1)$

Properties of STM

4. $$\varphi(\tau, t) = [\varphi(t, \tau)]^{-1}$$

5. For time-invariant systems

$$\varphi(0) = I$$

$$\varphi(t)\varphi(\tau) = \varphi(t + \tau)$$

$$\varphi^{-1}(t) = \varphi(-t)$$

6. For linear time invariant system

$$\varphi(t, \tau) = e^{A(t-\tau)}$$

Solution of Linear Time Varying Systems

Solution: $X(t) = \varphi(t, t_0)\, C(t)$ (Method of variation of parameters)

How to determine $C(t)$?

$$\dot{X} = AX + BU$$

$$\left[\frac{\partial \varphi}{\partial t} C\right] + \varphi\, \dot{C} = \left[A\varphi\, C\right] + BU \quad , \dot{C} = [\varphi(t, t_0)]^{-1} BU$$

$$C(t) = C(t_0) + \int_{t_0}^{t} \left[\varphi^{-1}(\tau, t_0)\right] B(\tau)\, U(\tau)\, d\tau$$

$$X(t_0) = C(t_0), \qquad \varphi(t_0, t_0) = I$$

Solution of Linear Time Varying Systems

$$X(t) = \varphi(t,t_0)\left[X(t_0) + \int_{t_0}^{t} \varphi^{-1}(\tau,t_0)\, B(\tau)\, U(\tau)\, d\tau \right]$$

$$= \varphi(t,t_0) X(t_0) + \int_{t_0}^{t} \left[\varphi(t,t_0)\varphi(t_0,\tau)\right] B(\tau)\, U(\tau)\, d\tau$$

$$X(t) = \varphi(t,t_0)\, X(t_0) + \int_{t_0}^{t} \varphi(t,\tau)\, B(\tau)\, U(\tau)\, d\tau$$

Thanks for the Attention….!