Lecture – 2

First and Second Order Linear Differential Equations

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Linear Differential Equations

Principle of superposition:

If $x_1(t)$ and $x_2(t)$ are solutions then $\alpha_1 x_1(t) + \alpha_2 x_2(t)$ is also a solution for any scalars α_1 and α_2

Example:

$$\ddot{x} + x = 1 \quad \text{(Non homogeneous)}$$

$x_1 = 1 + \cos t$ and $x_2 = 1 + \sin t$ both are solutions.

But

$2(1+\cos t)$ or $[(1+\cos t) + (1+\sin t)]$ are not solutions.

Linear Differential Equations

However, if

$$\ddot{x} + \dot{x} = 0 \quad \text{(Homogeneous)}$$

then both of the above are solutions.

<u>Lesson:</u> Principle of superposition holds good only for homogeneous linear differential equations.

Homogeneous First-order Equations

System dynamics: $\dot{x} + kx = 0, \quad x(t_0) = x_0, \quad k = 1/\tau$

Solution:

$$\left(dx/x\right) = -k\,dt$$

$$\ln x = -k\,t + \ln c$$

$$\ln\left(x/c\right) = -k\,t$$

$$x = e^{-k\,t}c$$

Initial condition: $x_0 = e^{-k t_0} c, \quad c = e^{k t_0} x_0$

Hence, $$x(t) = e^{-k(t-t_0)} x_0$$

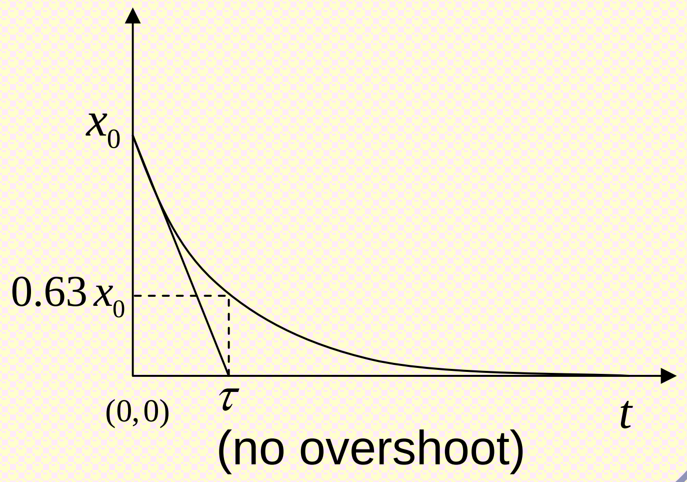
Homogeneous First-order Equations

Note:

1) $$e^{at} = 1 + at + \frac{a^2t^2}{2!} + \frac{a^3t^3}{3!} + \cdots$$

2) If $t_0 = 0$, then the solution is

$$x(t) = e^{-kt}x_0 = e^{-(t/\tau)}x_0$$



Non-homogeneous First-order Equations: Response to Step Inputs

Consider: $\dot{x} + \frac{1}{\tau} x = A \qquad (A\text{: constant})$

Homogeneous solution:

$$\dot{x}_h + \frac{1}{\tau} x_h = 0$$

$$x_h = e^{-\frac{t}{\tau}} c$$

Particular solution: $x_p = B$

Substitute: $\frac{1}{\tau} B = A$

$$\therefore x_p = B = \tau A$$

Non-homogeneous First-order Equations: Response to Step Inputs

$$x(t) = x_h + x_p$$

$$x(t) = e^{-\frac{t}{\tau}} c + \tau A$$

$$x_0 = c + \tau A \qquad \Rightarrow c = x_0 - \tau A$$

$$x(t) = e^{-\frac{t}{\tau}} \left(x_0 - \tau A \right) + \tau A$$

$$x_{ss} = \lim_{t \to \infty} x(t) = \tau A \qquad (\text{Note: } x_{ss} \neq A)$$

Complete solution: $x(t) = e^{-\frac{t}{\tau}} \left(x_0 - \tau A \right) + \tau A$

Steady state error (*wrt*. input A):

$$e_{ss} = A - \left[x(t) \right]_{t \to \infty} = A - \tau A = \left(1 - \tau \right) A$$

First-order Equations in Special Form: Unit Step Response

Consider: $\dot{x} + \left(\frac{1}{\tau}\right)x = \left(\frac{1}{\tau}\right)A$ $\quad$ (Special case in text books: $A = 1$)

Homogeneous:

$$\dot{x}_h + \frac{1}{\tau}x_h = 0$$

$$x_h = e^{-\frac{t}{\tau}}c$$

Particular solution: $x_p = B$

Substitute: $\left(\frac{1}{\tau}B = \frac{1}{\tau}A\right) \Rightarrow (B = A)$

$$\therefore x_p = B = A$$

First-order Equations in Special Form: Unit Step Response

$$x(t) = x_h + x_p$$

$$x(t) = e^{-\frac{t}{\tau}} c + A$$

$$x_0 = c + A \quad \Rightarrow \quad c = x_0 - A$$

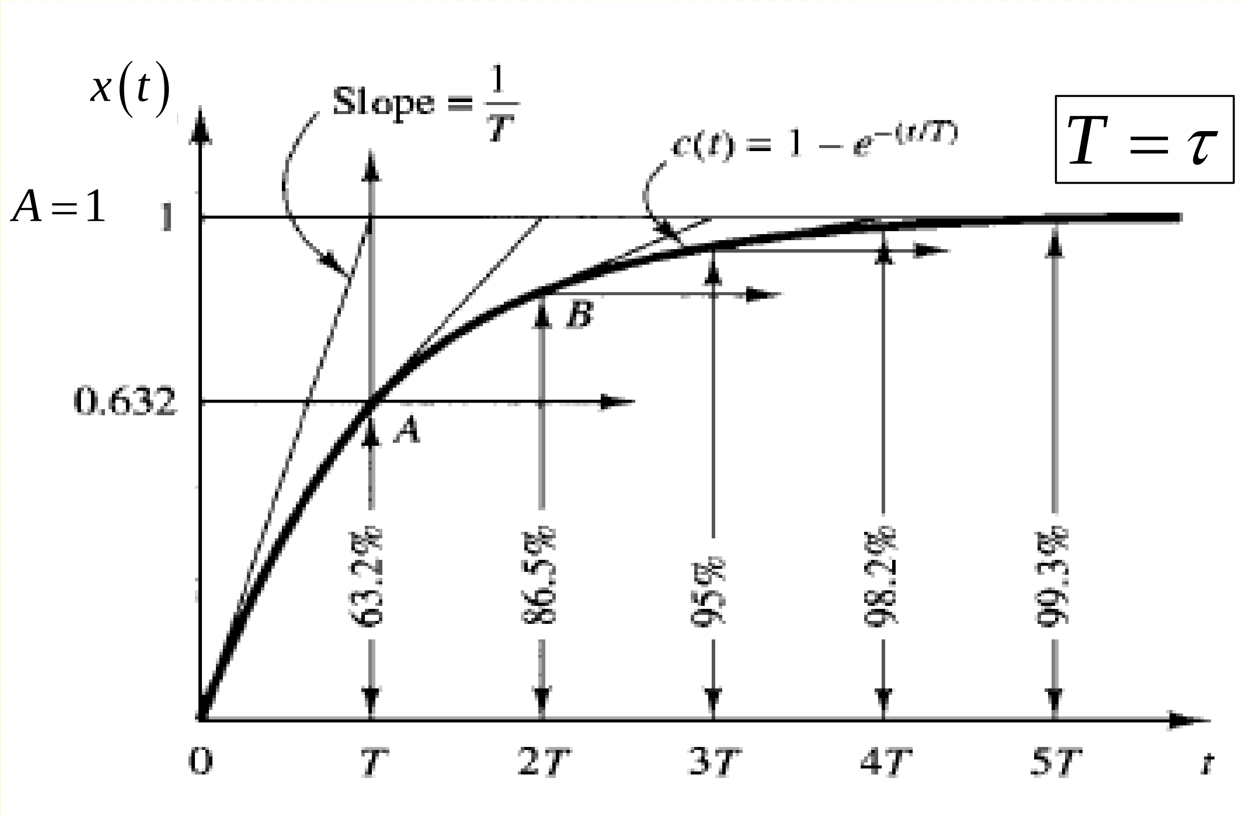
$$x(t) = e^{-\frac{t}{\tau}} \left(x_0 - A \right) + A$$

$$x_{ss} = \lim_{t \to \infty} x(t) = A$$

Complete solution: $x(t) = e^{-\frac{t}{\tau}} \left(x_0 - A \right) + A$

Steady state error (*wrt*. input A): $e_{ss} = A - x_{ss} = A - A = 0$

Unit Step Response of a First-Order System



Settling time: $T_s = 4T = 4\tau$

Second Order System

Homogeneous:

$$\ddot{x} + a\dot{x} + bx = 0, \qquad (a = 2\xi\omega_n, b = \omega_n{}^2)$$

Guess: $x(t) = e^{\lambda t}$

$$(\lambda^2 + a\lambda + b)e^{\lambda t} = 0$$

$$e^{\lambda t} \neq 0$$

$$\Rightarrow \lambda^2 + a\lambda + b = 0 \qquad \text{: characteristic equation}$$

$$\lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

Second Order System

Note: $a, b \in \mathbb{R}$

$\Rightarrow$ Arises three cases

Case 1: Distinct and real roots

Case 2: Complex conjugate roots

Case 3: Real and repeated roots

Case-1: Distinct and real roots

$$x_1 = e^{\lambda_1 t}, \qquad x_2 = e^{\lambda_2 t}$$

$$x(t) = c_1 x_1(t) + c_2 x_2(t)$$

$$\boxed{x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}}$$

Case-2: Complex conjugate roots

$$\lambda_1 = \sigma + j\omega, \qquad \lambda_2 = \sigma - j\omega$$

$$x_1 = e^{(\sigma + j\omega)t} = e^{\sigma t}(\cos\omega t + j\sin\omega t)$$

$$x_2 = e^{(\sigma - j\omega)t} = e^{\sigma t}(\cos\omega t - j\sin\omega t)$$

$$x(t) = c_1 e^{(\sigma + j\omega)t} + c_2 e^{(\sigma - j\omega)t}$$

However notice that:

Linearly independent terms

$$\frac{x_1 + x_2}{2} = e^{\sigma t}\cos\omega t$$

$$\frac{x_1 - x_2}{2j} = e^{\sigma t}\sin\omega t$$

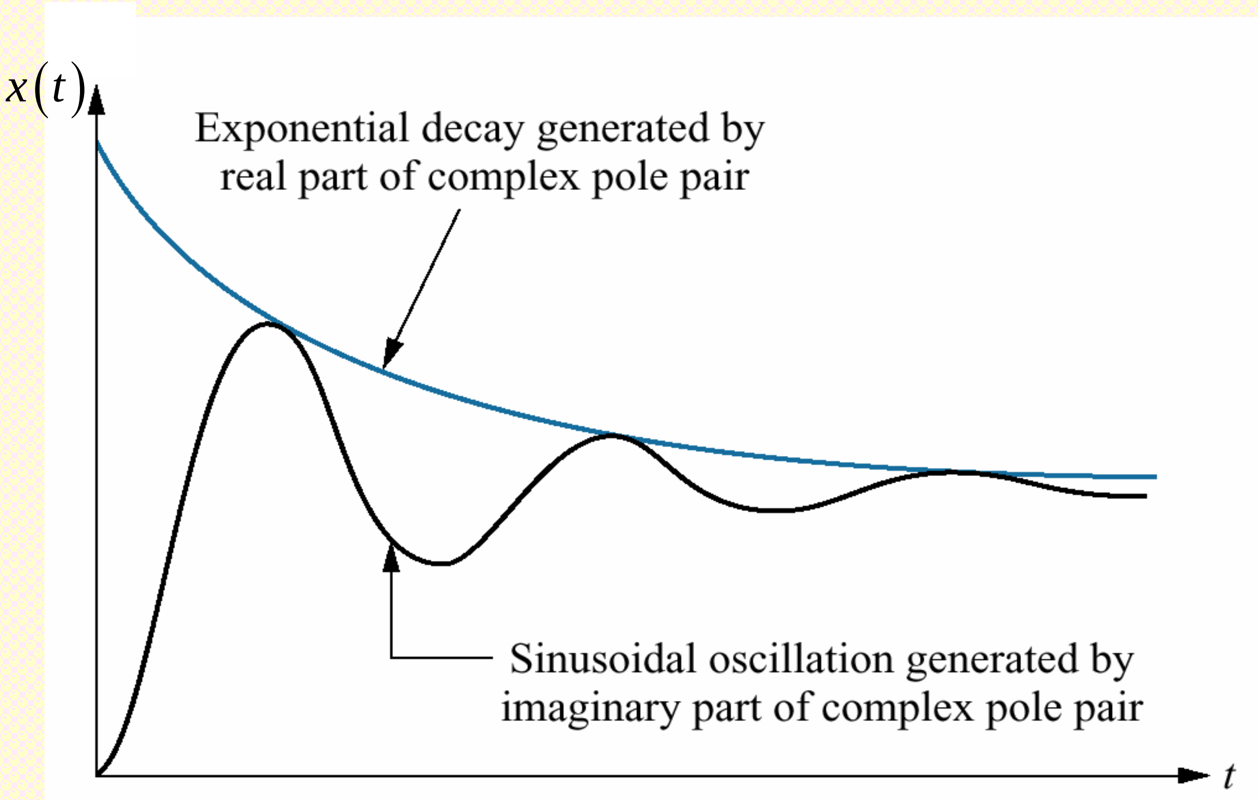
Case-2: Complex conjugate roots

So, we can also write the solution as :

$$x(t) = A\left(\frac{x_1 + x_2}{2}\right) + B\left(\frac{x_1 - x_2}{2j}\right)$$

$$\boxed{x(t) = e^{\sigma t}\left(A\cos\omega t + B\sin\omega t\right)}$$

Complex conjugate roots: Typical response plot



Case-3: Real and repeated roots

$\lambda_1 = \lambda_2 = \lambda$

$x_1(t) = e^{\lambda t}$

How to find $x_2(t)$

Method of variation of parameters

Assume: $x_2(t) = u(t)\; x_1(t)$

Then

$$\dot{x}_2 = u\,\dot{x}_1 + \dot{u}\,x_1$$

$$\ddot{x}_2 = u\,\ddot{x}_1 + \dot{u}\,\dot{x}_1 + \dot{u}\,\dot{x}_1 + \ddot{u}\,x_1$$

Case-3: Real and repeated roots

Substituting back and rearranging the terms, it leads to

$$u\left(\ddot{x}_1 + a\,\dot{x}_1 + b\right) + \dot{u}\left(2\dot{x}_1 + a\,x_1\right) + \ddot{u}\,x_1 = 0$$

However, $\left(\ddot{x}_1 + a\,\dot{x}_1 + b\right) = 0 \qquad \left(\text{since } x_1 \text{ is a solution}\right)$

Moreover, $\dot{x}_1 = \lambda e^{\lambda t} = \lambda\,\dot{x}_1 = -\left(\dfrac{a}{2}\right)x_1 \qquad \left(\text{since } \lambda = -\dfrac{a}{2}\right)$

$$\left(2\dot{x}_1 + a\,x_1\right) = 0$$

This leads to $\ddot{u}\,x_1 = 0$. Moreover $x_1 = e^{\lambda t} \neq 0$. Hence $\ddot{u} = 0$

Case-3: Real and repeated roots

One solution $u(t) = t$

$$\therefore x_2(t) = t\, x_1(t) = te^{\lambda t}$$

$$x(t) = c_1 x_1(t) + c_2 x_2(t)$$

$$\boxed{x(t) = (c_1 + c_2 t)\, e^{\lambda t}}$$

Corollary: When $\lambda = 0$ (double poles at origin)

$$x(t) = c_1 + c_2 t \to \infty \quad \text{as} \quad t \to \infty$$

Double pole (in general, multiple poles) at origin is de-stabilizing..!

Summary

Case	Roots of characteristic equation	Basics	General Solution
1	Distinct & Real λ_1, λ_2	$e^{\lambda_1 t}, e^{\lambda_2 t}$	$c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$
2	Complex Conjugate $\lambda = \sigma + j\omega$	$e^{\sigma t} \cos \omega t$ $e^{\sigma t} \sin \omega t$	$e^{\sigma t}(A \cos \omega t + B \sin \omega t)$
3	Real double Roots	$e^{\lambda t}, te^{\lambda t}$	$(c_1 + c_2 t)e^{\lambda t}$

Example

$\ddot{x} - 4\dot{x} + 4x = 0 \qquad x(0) = 3, \quad \dot{x}(0) = 1$

$\lambda^2 - 4\lambda + 4 = 0 \qquad \lambda = 2, 2$ (Real, repeated)

$$x(t) = (c_1 + c_2 t) e^{2t}$$

$$\dot{x}(t) = 2(c_1 + c_2 t) e^{2t} + c_2 e^{2t}$$

Solving for the initial conditions $\quad c_1 = 3 \qquad c_2 = -5$

$$x(t) = (3 - 5t) e^{2t}$$

Particular Solution: Method of Undetermined Coefficients

$$\ddot{x} + a\dot{x} + bx = f(t)$$

Terms in $f(t)$	Choice of $x_p(t)$
$k\,e^{pt}$	$c\,e^{pt}$
$k\,t^{n}$	$k_n\,t^n + k_{n-1}\,t^{n-1} + \cdots + k_1 t + k_0$
$k\cos\omega t$	$A\cos\omega t + B\sin\omega t$
$k\sin\omega t$	$A\cos\omega t + B\sin\omega t$

Substitute these expressions back in the differential equation and determine the unknown coefficients

Example

Find $x_p(t)$ for $\ddot{x} - 3\dot{x} + 2x = 4t + e^{3t}$

Choose $x_p(t) = (k_1 t + k_0) + ce^{3t}$

Substitute and equate the coefficients

$$k_1 = 2, \quad k_0 = 3, \quad c = \frac{1}{2}$$

$$\therefore \quad x_p(t) = (2t + 3) + \frac{1}{2} e^{3t}$$

Second order system in Standard form

$$\ddot{c} + 2\varsigma\omega_n \dot{c} + \omega_n^2 c = \omega_n^2 u$$

ω_n : Natural frequency

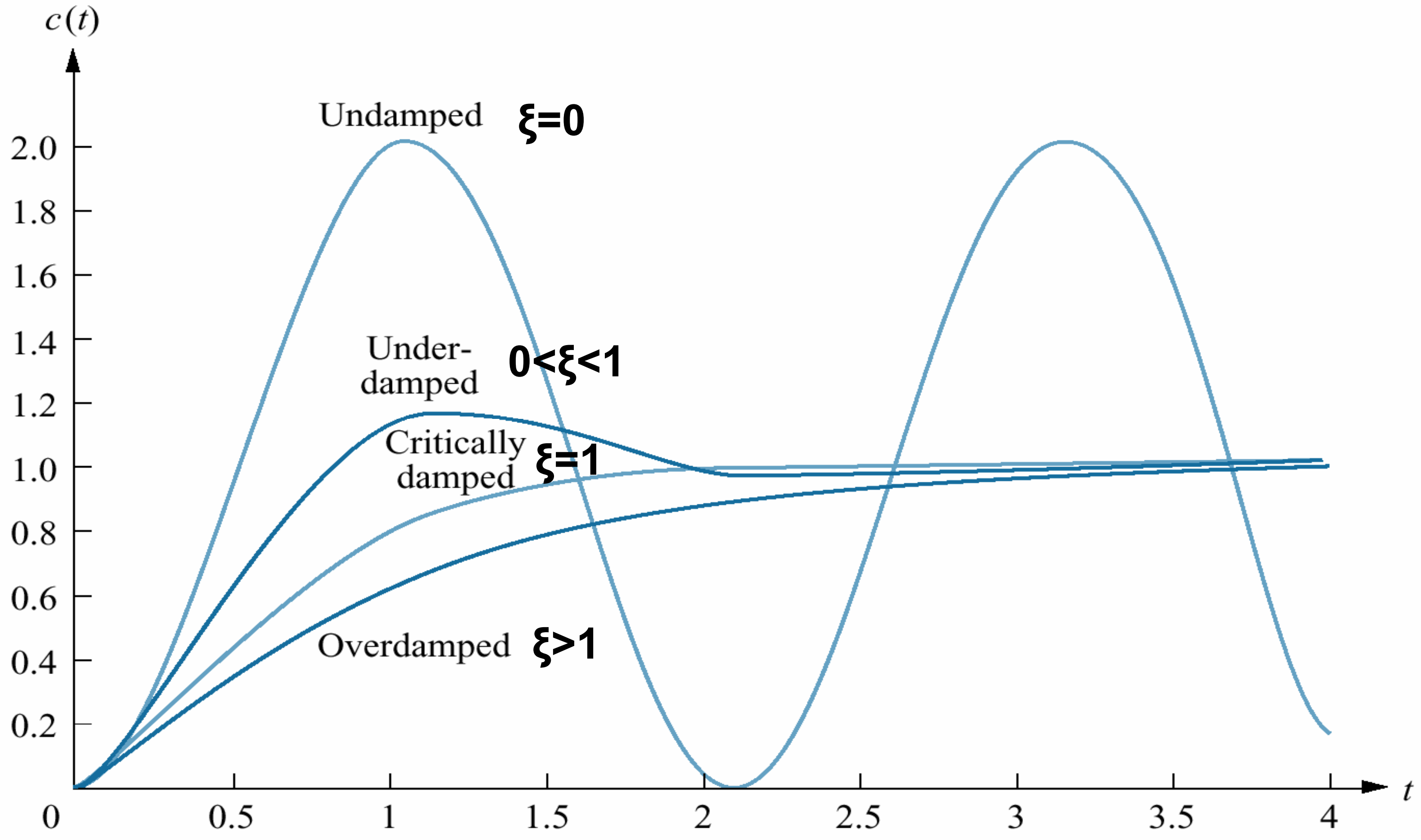
ς : Damping ratio

Transfer function (will be studied later):

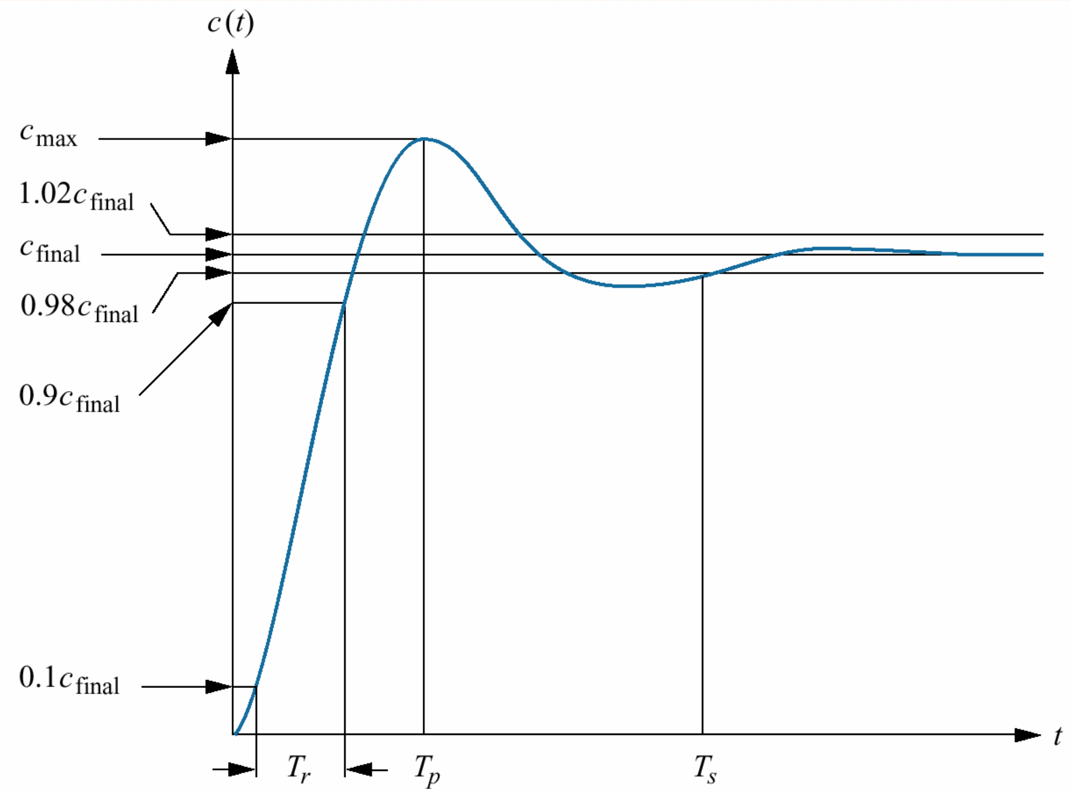
$$\frac{C(s)}{U(s)} = \frac{\omega_n^2}{s^2 + 2\varsigma\omega_n s + \omega_n^2}$$

Roots: poles

Unit Step Response of a Second-order System



Under-damped system response specifications



Ref: N. S. Nise:
Control Systems Engineering,
4th Ed., Wiley, 2004

T_r : Rise time

T_p : Pick time

T_s : Settling time

Transient Response Specifications

Rise time: $T_r = \dfrac{\pi - \beta}{\omega_d}$, Peak time: $T_p = \dfrac{\pi}{\omega_d}$

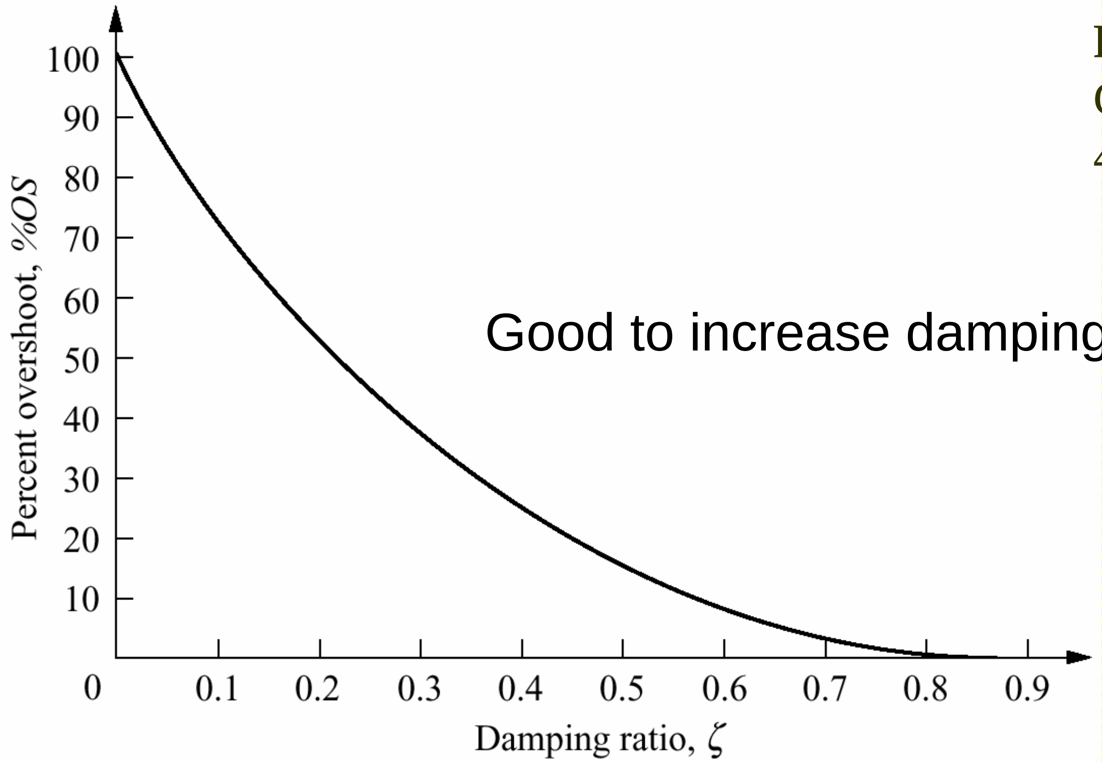
Damped natural frequency: $\omega_d = \omega_n \sqrt{1 - \zeta^2}$

Maximum over shoot: $M_p = e^{\left(-\pi\xi \big/ \sqrt{1-\zeta^2} \right)}$

Settling time: $T_s = \dfrac{4}{\xi\omega_n} \quad (2\%\ criterion)$

$$= \dfrac{3}{\xi\omega_n} \quad (5\%\ criterion)$$

Percentage over-shoot as a function of damping ratio



Good to increase damping ratio.

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Control Systems Engineering,
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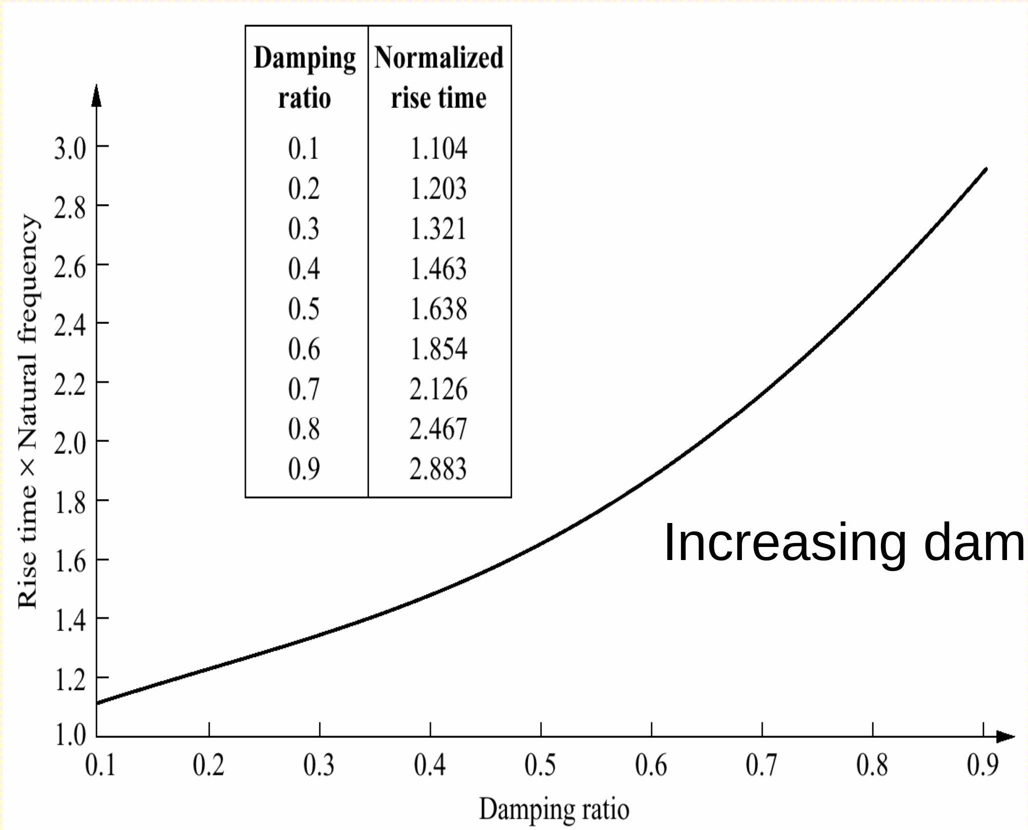
Rise time as a function of damping ratio



Damping ratio	Normalized rise time
0.1	1.104
0.2	1.203
0.3	1.321
0.4	1.463
0.5	1.638
0.6	1.854
0.7	2.126
0.8	2.467
0.9	2.883

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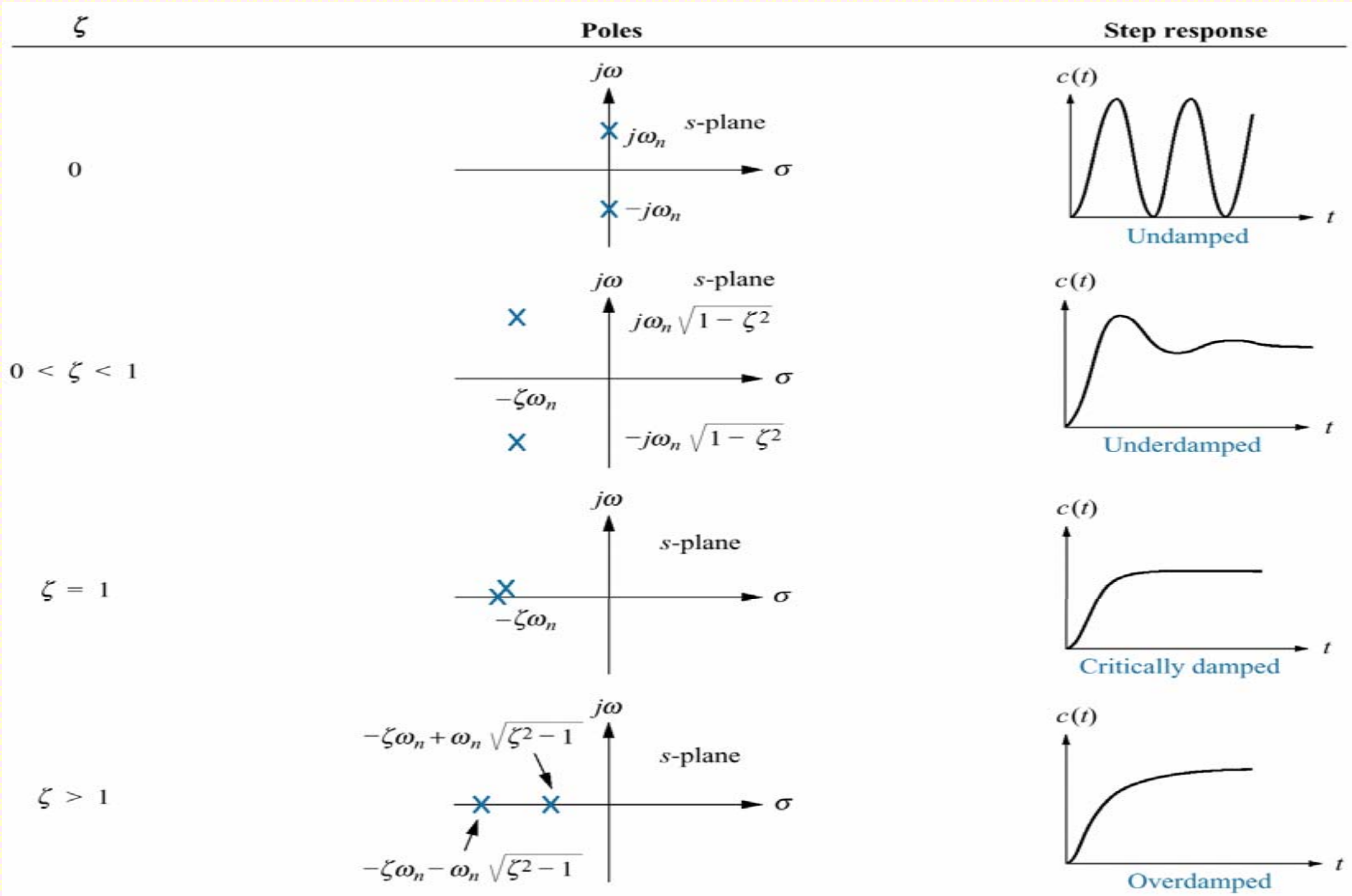
Increasing damping ratio too much is Bad..!

Second-order System Response as a Function of Damping Ratio



Thanks for the Attention...!