<u>Lecture – 16</u>

Review of Numerical Methods

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Linear Equations: Solution Technique

$$AX = b \qquad A \text{ is nonsingular, } b \neq 0$$

Problem: $X = ?$

Motivation: $\dot{X} = AX + BU$

where $X = \left[\begin{array}{c} X_c \\ \hline X_N \end{array}\right]$

X_C : controlled state

X_N : uncontrolled state

$\dim(X_C) = \dim(U) = m$

$$\begin{bmatrix} \dot{X}_C \\ \dot{X}_N \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} X + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U$$

Motivation: Continued

$$\cancel{\dot{X}_c} = A_1 X + B_1 U$$

$$\mathbf{0}$$

Which gives

$$U = -B_1^{-1}\left(A_1 X\right)$$

Note: B_1 is square

This will be the controller necessary to maintain X_c at steady state

Solution Technique: Direct Inversion of **A**

$$X = A^{-1}b$$

- Computation of A^{-1} Involves too many computations, roughly $n^2 \times n!$ number of operations (very inefficient for large n).
- This approach also suffers from the problem of sensitivity (ill-conditioning) ,when $|A| \to 0$
- Round off errors may lead to large inaccuracies

Solution Technique: Gauss Elimination

- Do row operations to reduce the A matrix to a upper triangular form
- Solve the variable from down to top

Example:

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

Solution Steps:

Step-I: Multiply row-1 with -1/2 and add to the row-2. row-3 keep unchanged, since $a_{31}=0$.

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3/2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3/2 \\ 4 \end{bmatrix}$$

Solution Technique: Gauss Elimination

Step-II: Multiply row-2 with -2/3 and add to row-3

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3/2 & 1 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3/2 \\ 3 \end{bmatrix}$$

Upper Triangle Matrix

Final Solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 9 \end{bmatrix}$$

Gauss Elimination

- The total number of operations needed is $(2/3)n^3$ which is far lesser than computing $A^{-1} = \dfrac{adj(A)}{|A|}$ (which requires $n^2 \times n!$ operations)

- The Gauss elimination method will encounter potential problems when the pivot elements i.e.. diagonal elements become zero, or very close to zero at any stage of elimination.
- In such cases the order of equations can be changed by exchanging rows and the procedure can be continued

Nonlinear Algebraic Equations

Problem: $F(X) = 0 \qquad X = ?$

Motivation: Finding the forced equilibrium condition for a nonlinear system to get an appropriate operating point for linearization

$$\dot{X} = f(X, U)$$

$$\left[\frac{\dot{X}_C}{X_N}\right] = \left[\frac{f_C(X,U)}{f_N(X,U)}\right] \qquad \dim(X_C) = \dim(f_C) = \dim(U)$$

$$\cancelto{0}{\dot{X}_C} = f_C(X_0, U_0)$$

Solve for U_0 from $f_C(U) = 0$

Newton-Raphson Method: Scalar Case

$$f(x) = 0$$

Using Taylor series expansion

$$\cancelto{0}{f\left(x_k + \Delta x_k\right)} \approx f\left(x_k\right) + \left[\frac{df}{dx}\right]_{x_k} \Delta x_k + \cdots + (higher\ order\ terms)$$

$$\left[\frac{df}{dx}\right]_{x_k} \Delta x_k = -f\left(x_k\right)$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

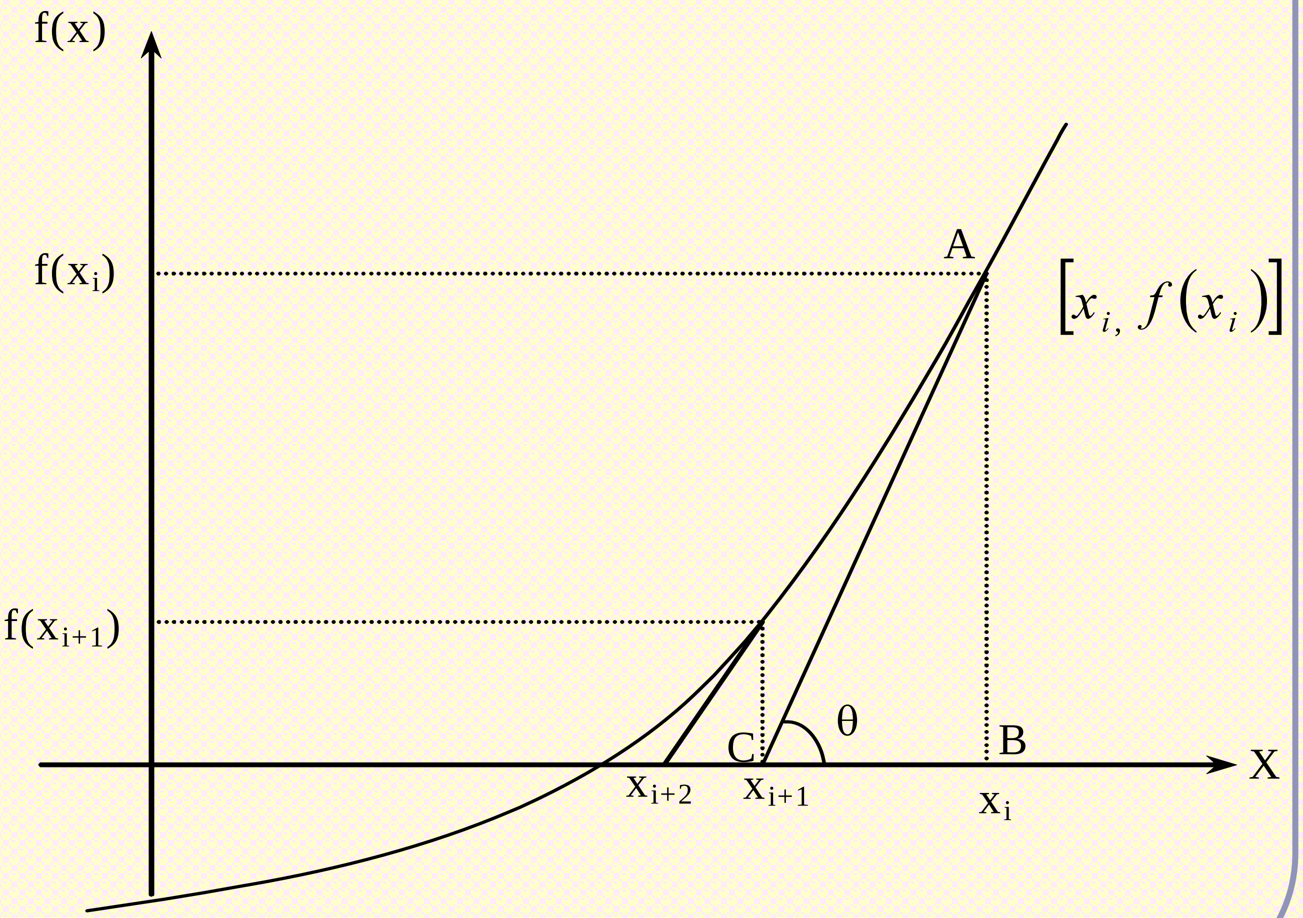
From the above equation with an initial guess

we can iteratively solve for x with $\Delta x <$ *tolerence*

Newton-Raphson Method: Scalar Case

$$f'(x_i) = \frac{f(x_i)}{x_i - x_{i+1}}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$



Newton-Raphson Method: Multi Variable Case

$$F(X) = 0$$

$$F\left(X_k + \Delta X_k\right) \approx F\left(X_k\right) + \overbrace{\left[\frac{\partial F}{\partial X}\right]_k}^{A_k} \Delta X_k$$

$$A_k \; \Delta X_k = -F\left(X_k\right)$$

Solve for $\quad \Delta X_k = -A_k^{-1} F\left(X_k\right)$

Update $\quad X_{k+1} = X_k + \Delta X_k$

Newton-Raphson Method: Algorithm

- Start with guess value x_1
- Solve for Δx_k
- Update $x_{k+1} = x_k + \Delta x_k \quad (k = 1, 2, \ldots)$
- Continue until convergence

Convergence Condition

1. Relative Error

$$\in_{a_k} \triangleq \left| \left(x_{k_{i+1}} - x_{k_i} \right) / x_{k_{i+1}} \right| < \text{tol}, \quad \forall k$$

2. Absolute Error

$$\left\| f\left(x_k\right) \right\| < \text{tol}$$

Example: N-R Method

Question : Find a root of the following equation

$$f(x) = x^3 - 0.165x^2 + 3.993\text{x}10^{-4} = 0$$

Solution : $f'(x) = 3x^2 - 0.33x$. Let $x_0 = 0.02$. Then

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.02 - \frac{3.413\text{x}10^{-4}}{-5.4\text{x}10^{-3}} = 0.08320$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.08320 - \frac{-1.670\text{x}10^{-4}}{-6.689\text{x}10^{-3}} = 0.05824$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.05284 - \frac{3.717\text{x}10^{-5}}{-9.043\text{x}10^{-3}} = 0.06235$$

Newton-Raphson Method: Advantages

- If it converges, it converges fast!

  It has "Quadratic convergence" property, i.e.

  $$e_{k+1} = c\, e_k^{\,2}, \quad \text{where} \quad e_k \triangleq \left(x^* - x_k \right)$$

  c is a constant

  x^* is the actual root

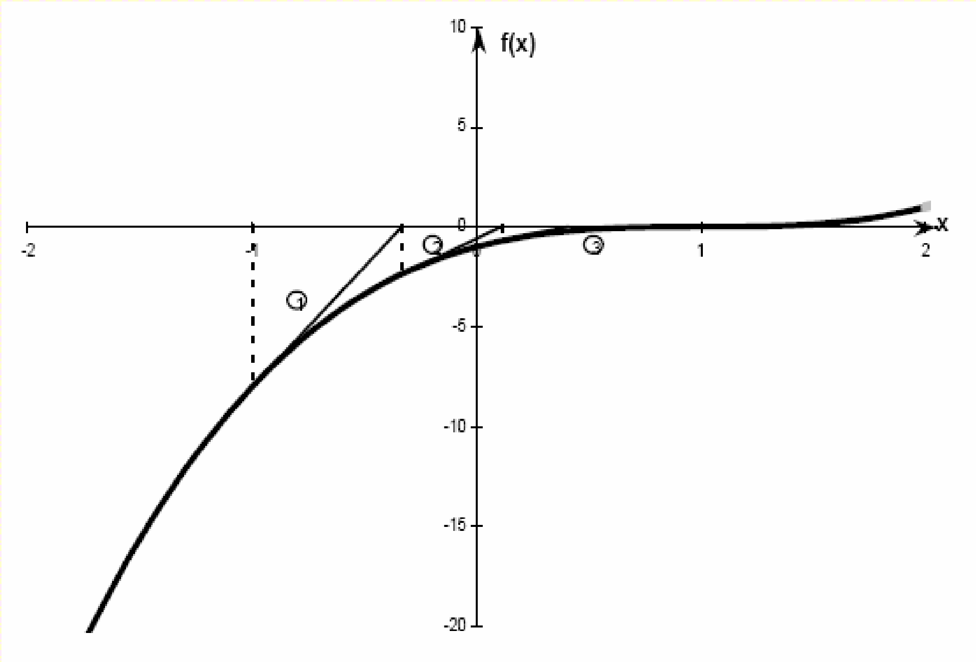
- Problem: It requires good initial guess in general to converge to the right solution.

Newton-Raphson Method: Limitations

- ***Non-convergence at Inflection points***

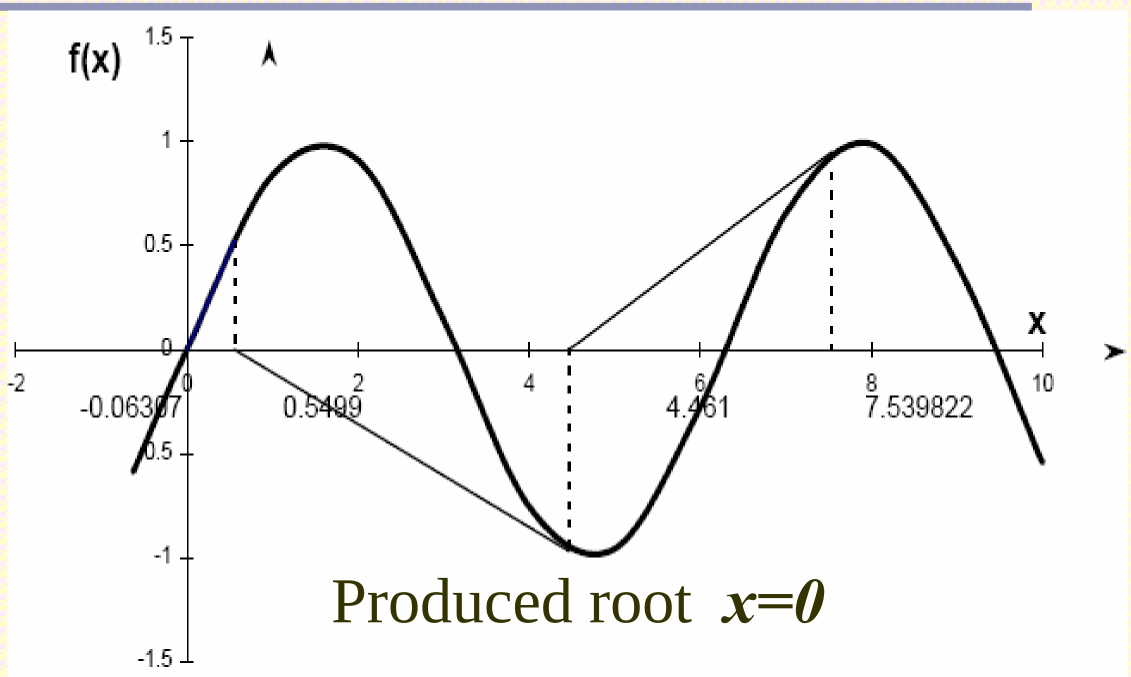
For a function f(x) the points where the concavity changes from up-to-down or down-to-up are called *inflection points.*

$$f(x) = (x-1)^3 = 0$$



Newton-Raphson Method: Limitations

- ***<u>Root Jumping</u>***



Cases where ***f (x)*** is oscillating and has a number of roots

Initial Guess near to one root may produce another root

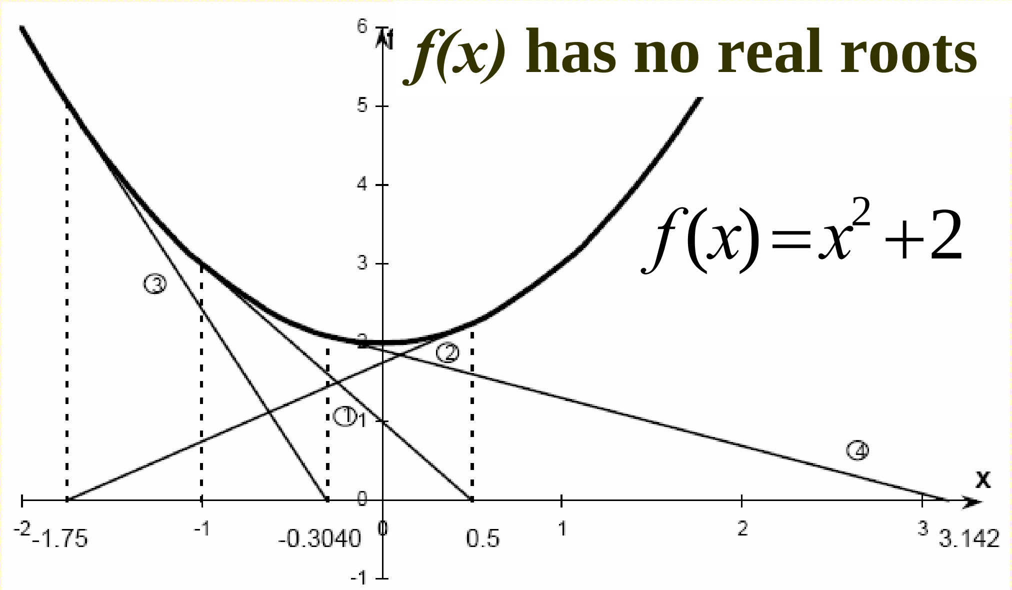
Example:

$$f(x) = \sin(x) = 0$$

Newton-Raphson Method: Limitations

- ***Oscillations around local minima or maxima***

  Results may oscillate about the local maximum or minimum without converging on a root but converging on the local maximum or minimum. Eventually, it may lead to division to a number close to zero and may diverge.



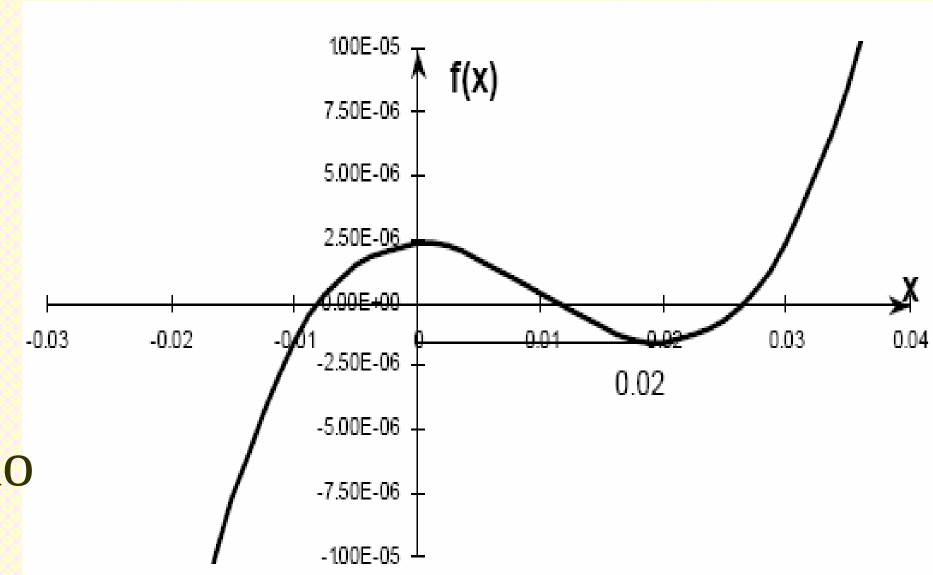
Newton-Raphson Method: Limitations

- ***Division by zero***

If $f'(x_i) \approx 0$ at some x_i, *x_{i+1} becomes very large value*

$$f(x) = x^3 - 0.03x^2 + 2.4\text{x}10^{-6} = 0$$

$$f'(x) = 3x^2 - 0.06x$$



Even after several iterations there is no convergence!

N-R Method Drawbacks

- ***$f'(x^*)$ is unbounded***

If the derivative of $f(x)$ *is unbounded* at the root then Newton-Raphson method will not converge.

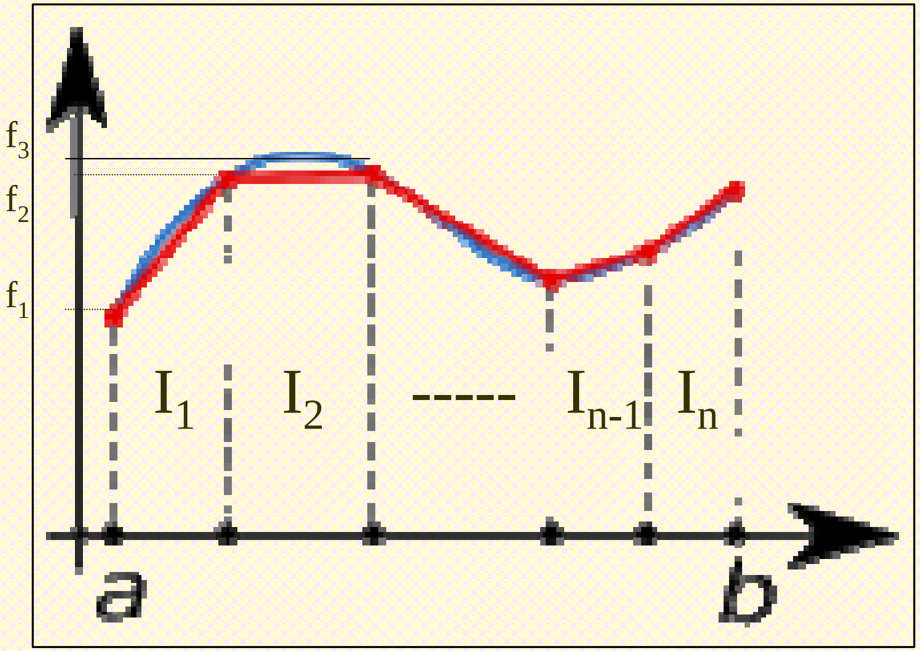
Exercise: Verify for $f(x)=\sqrt{x}$

Numerical Differentiation $\left(\frac{df}{dx}\right)$

Technique Name	Definition	Numerical Approximation	Error
Forward difference	$\lim_{\Delta x\to 0}\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]$	$\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$	$O(\Delta x)$
Backward difference	$\lim_{\Delta x\to 0}\left[\frac{f(x)-f(x-\Delta x)}{\Delta x}\right]$	$\frac{f(x_0)-f(x_0-\Delta x)}{\Delta x}$	$O(\Delta x)$
Central difference	$\lim_{\Delta x\to 0}\left[\frac{f(x+\Delta x)-f(x-\Delta x)}{2\Delta x}\right]$	$\frac{f(x_0+\Delta x)-f(x_0-\Delta x)}{2\Delta x}$	$O(\Delta x^2)$

Numerical Integration

Trapezoidal Rule:



Note: *Numerical Differentiation is "Error Amplifying".*

where as Numerical Integration is "Error Smoothing".

Numerical Integration

- Trapezoidal Rule:

$$
\begin{aligned}
I &\approx I_1 + I_2 + \cdots + I_{n\text{-}1} + I_n \\
&= \frac{1}{2}\Delta x\left(f_0 + f_1\right) + \frac{1}{2}\Delta x\left(f_1 + f_2\right) + \cdots \\
&\qquad + \frac{1}{2}\Delta x\left(f_{n-2} + f_{n-1}\right) + \frac{1}{2}\Delta x\left(f_{n-1} + f_n\right) \\
&= \frac{\Delta x}{2}\left[f_0 + 2f_1 + 2f_2 + \cdots + 2f_{n-1} + f_n\right]
\end{aligned}
$$

$$
\mathrm{I} \approx \left(\frac{f_0}{2} + f_1 + \cdots + f_{n-1} + \frac{f_n}{2}\right)\Delta x
$$

Ordinary Differential Equation (ODE)

$$f\left(x, \frac{dx(t)}{dt}, \frac{d^2x(t)}{dt^2}, \cdots \frac{d^nx(t)}{dt^n}\right) = 0$$

- **Ordinary**: only one independent variable
- **Differential Equation**: unknown functions enter into the equation through its derivatives
- **Order**: highest derivative in f
- **Degree**: exponent of the highest derivative

$$Example: \quad \left(\frac{d^3x(t)}{dt^3}\right)^4 - x(t) = 0$$

$$\text{degree} = 4; \ \text{order} = 3$$

What Is Solution of ODE ??

- A problem involving ODE is not completely specified by its equation

  ODE has to be supplemented with **boundary conditions.**

- **Initial value problem**: $\boldsymbol{x}$ is given at some starting value $\boldsymbol{t_i}$, and it is desired to find at some final points $\boldsymbol{t_f}$ or at some discrete list of points.
- **Two point boundary value problem**: Boundary conditions are specified at more than one $\boldsymbol{t}$; typically some of the conditions will be specified at $\boldsymbol{t_i}$ and some at $\boldsymbol{t_f}$.

Numerical Solution to Initial Value Problem

$$\frac{dx(t)}{dt} = f(t, x(t)); \quad x(t_0) = x_0$$

- A numerical solution to this problem generates sequence of values for the independent variable $t_1, t_2, \ldots t_n$ and a corresponding sequence of values of the dependent variable $x_1, x_2, \ldots, x_n$ so that each x_n approximates solution at t_n

$$\boldsymbol{x_n \approx x(t_n) \quad n = 0, 1, 2 \ldots . n.}$$

Basic Concepts of Numerical Methods to Solve ODEs

$$\frac{x_{n+1} - x_n}{\Delta t} \approx \text{ slope of tangent}$$

We can calculate the **tangent slope** at any point. In fact the differential equation

$$\frac{dx(t)}{dt} = f\left(t, x(t)\right) \text{ defines the}$$

$$\text{tangent slope } = f\left(t, x(t)\right)$$

Euler's Method

- Solve $\dfrac{dx}{dt} = f(t, x)$ with $x(0) = b$
- At start of time step

$$\frac{x_{n+1} - x_n}{\Delta t} \approx f(t_n, x_n) \quad \text{Forward difference}$$

Rearranging

$$x_{n+1} = x_n + \Delta t \; f_n$$

Start with initial conditions $t_0 = 0$; $x_0 = b$

Euler Integration: Useful Comments

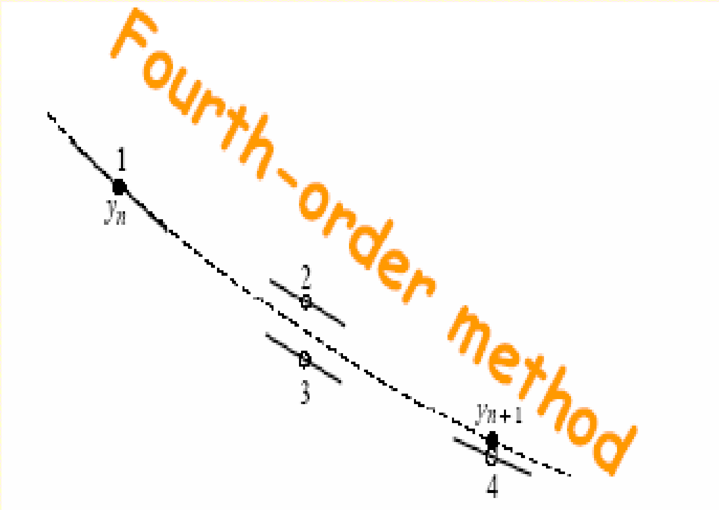
- Euler integration has error of the order of $(\Delta t)^2$

- Small step size Δt may be needed for good accuracy. This is in conflict with the computational load advantage.

- Lesser computational load

Runge-Kutta Fourth Order Method

$$x_{i+1} = x_i + \frac{\Delta t}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$



where

$$k_1 = f\left(t_i, x_i\right)$$

$$k_2 = f\left(t_i + \frac{1}{2}\Delta t, x_i + \frac{1}{2}k_1\Delta t\right)$$

$$k_3 = f\left(t_i + \frac{1}{2}\Delta t, x_i + \frac{1}{2}k_2\Delta t\right)$$

$$k_4 = f\left(t_i + \Delta t, x_i + k_3\Delta t\right)$$

In each step the derivative is calculated at four points, once at the initial point, twice at trial mid points and once at trial end point

Runge-Kutta Algorithm

- Error $\theta\left(\{\Delta t\}^5\right)$

- The method uses a 4th order power series approximation to come up with this algorithm. Hence, the algorithm is called RK-4 method

Thanks for the Attention….!