Lecture – 13

Review of Matrix Theory – I

Dr. Radhakant Padhi

Asst. Professor

Dept. of Aerospace Engineering

Indian Institute of Science - Bangalore

Definition and Examples

- **Definition:** Matrix is a collection of elements (numbers) arranged in rows and columns.
- **Examples:**

$$A_{1\times 3} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \quad B_{3\times 1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad C_{2\times 3} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix},$$

$$D_{3\times 2} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, \quad E_{3\times 3} = \begin{bmatrix} 1 & 4 & 6 \\ 4 & 2 & 5 \\ 6 & 5 & 3 \end{bmatrix}$$

Definitions

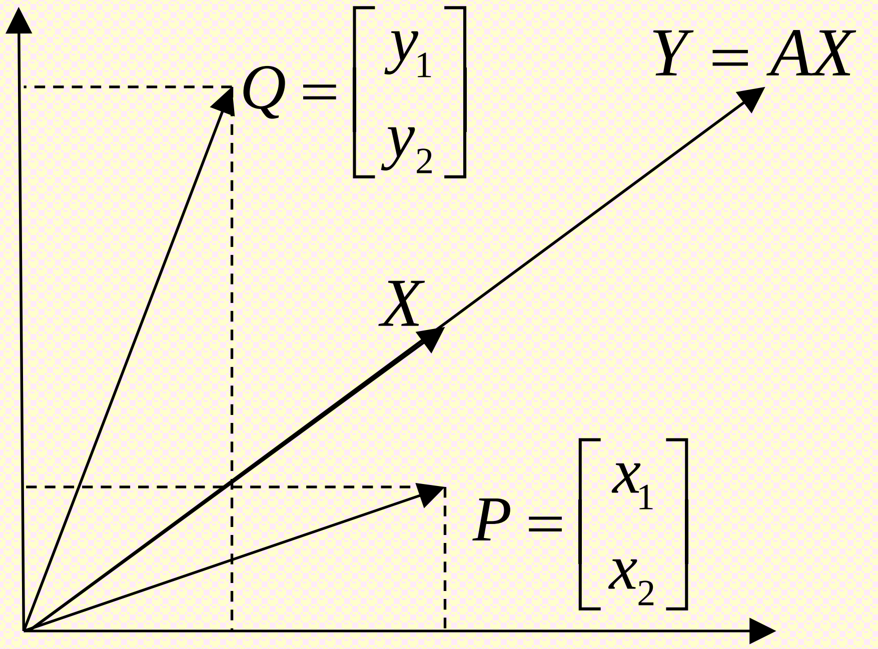
- Symmetric matrix: $A = A^T$
- Singular matrix: $|A| = 0$
- Inverse of a matrix: B is inverse of A **iff** $AB = BA = I$

  $A^{-1} = adj(A)/|A|$
- Orthogonal matrix: $AA^T = A^T A = I$
  - Example: $T(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
  - Result: Columns of an orthogonal matrix are orthonormal.

Eigenvalues and Eigenvectors

Matrices also act as linear operators with "stretching" and "rotation" operations.

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



Eigenvalues and Eigenvectors

- Question: Can we find a direction (vector), along which the matrix will act only as a stretching operator?
- Answer: If such a solution exists, then

$$AX = \lambda X \quad \Rightarrow \quad (\lambda I - A) X = 0$$

- For nontrivial solution, $|\lambda I - A| = 0$
- Utility: Stability and control, Model reduction, Principal component analysis etc.

Terminology	Definition	Properties of Eigenvalues
Positive definite $A > 0$	$X^T A X > 0 \quad \forall X \neq 0$	$\lambda_i > 0, \quad \forall i$
Positive semi definite $A \geq 0$	$X^T A X \geq 0 \quad \forall X \neq 0$	$\lambda_i \geq 0, \quad \forall i$
Negative definite $A < 0$	$X^T A X < 0 \quad \forall X \neq 0$	$\lambda_i < 0, \quad \forall i$
Negative semi definite $A \leq 0$	$X^T A X \leq 0 \quad \forall X \neq 0$	$\lambda_i \leq 0, \quad \forall i$

Eigenvalues and Eigenvectors: Some useful properties

- If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of $A_{n \times n}$ then for any positive integer *m*, $\lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m$ are eigenvalues of A^m

- If A is a nonsingular matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then $\lambda_1^{-1}, \lambda_2^{-1} \ldots\ldots\ldots \lambda_n^{-1}$ are eigenvalues of A^{-1}

- For triangular matrix, the eigenvalues are the **diagonal elements**

Eigenvalues and Eigenvectors: Some useful properties

- **If a $A_{n\times n}$ matrix is symmetric, its eigenvalues are all *REAL*. Moreover, it has n linearly-independent eigenvectors.**

- If $A_{n\times n}$ has *n* real eigenvalues and *n* real orthogonal eigenvectors, then the matrix is symmetric

- $A^T A$ and AA^T are always positive semi definite.

- If A is a positive definite symmetric matrix, then every principal sub-matrix of A is also symmetric and positive definite. In particular, the diagonal elements of A are positive.

Generalized Eigenvectors

If an eigenvalue is repeated *p* times, there may or may not be *p* linearly independent eigenvectors corresponding to it. In case linearly independent eigenvalues cannot be found, generalized eigenvectors are the next option.

<u>Example:</u>

Suppose $A_{3\times 3}$ has eigenvalues $\lambda_1, \lambda_2, \lambda_2$.

Then eigenvectors V_1, V_2 and generalized eigenvector V_3 can be found as follows:

(a) $\left(A - \lambda_1 I\right) V_1 = 0$

(b) $\left(A - \lambda_2 I\right) V_2 = 0$

(c) $\left(A - \lambda_3 I\right) V_3 = V_2$

Vector Norm

Vector norm is a "real valued function" with the following properties:

(a) $\|X\| > 0$ and $\|X\| = 0$ only if $X = 0$

(b) $\|\alpha X\| = |\alpha| \|X\|$

(c) $\| X + Y\| \leq \|X\| + \|Y\| \quad \forall X, Y$

Vector Norm

$$\left\|X\right\|_1 = |x_1| + |x_2| + \cdots + |x_n| \qquad (l_1 \text{ norm})$$

$$\left\|X\right\|_2 = \left(|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2\right)^{1/2} \qquad (l_2 \text{ norm})$$

$$\left\|X\right\|_3 = \left(|x_1|^3 + |x_2|^3 + \cdots + |x_n|^3\right)^{1/3} \qquad (l_3 \text{ norm})$$

$$\vdots$$

$$\left\|X\right\|_p = \left(|x_1|^p + |x_2|^p + \cdots + |x_n|^p\right)^{1/p} \qquad (l_p \text{ norm})$$

$$\vdots$$

$$\left\|X\right\|_\infty = \left(|x_1|^\infty + |x_2|^\infty + \cdots + |x_n|^\infty\right)^{1/\infty} = \max_i |x_i| \qquad (l_\infty \text{ norm})$$

Matrix/Operator/Induced Norm

Definition:

$$\|A\| = \max_{X \neq 0} \frac{\|A\,X\|}{\|X\|} = \max_{\|X\|=1} \left(\|A\,X\|\right)$$

Properties:

(a) $\|A\| > 0$ and $\|A\| = 0$ only if $A = 0$

(b) $\|\alpha\; A\| = |\alpha|\,\|A\|$

(c) $\|\;A + B\| \leq \;\|A\| + \|B\|$

(d) $\|\;AB\| \leq \;\|A\|\;\|B\|$

Matrix/Operator/Induced Norm

- 1-Norm

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}| : \quad \text{Largest of the absolute column sums}$$

- 2-Norm

$$\|A\|_2 = \sigma_{\max}(A) : \quad \text{Largest Singular Value}$$

- ∞ -Norm

$$\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}| : \quad \text{Largest of the absolute row sums}$$

Matrix/Operator/Induced Norm

- Frobenius Norm
  - Holds good for non-square matrices as well
  - Used frequently in neural network and adaptive control literature

$$\left\| A_{m\times n} \right\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2 \right)^{1/2}$$

Spectral Radius

For $A_{n\times n}$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ the spectral radius $\rho(A)$ is defined as

$$\rho(A) = \max_{1 \le i \le n} |\lambda_i|$$

<u>A Result:</u>

$$\|A\|_2 = \left[\rho\left(A^T A\right)\right]^{1/2}$$

If A is symmetric, then

$$\|A\|_2 = \left[\rho\left(A^T A\right)\right]^{1/2} = \left[\rho\left(A^2\right)\right]^{1/2} = \left[\left(\rho(A)\right)^2\right]^{1/2} = \rho(A)$$

Least Square Solutions

System: $AX = b$ where $A \in R^{m \times n}, \quad X \in R^{n}, \quad b \in R^{m}$

Case 1: $\left(m = n \text{ and } \left| A \right| \neq 0 \right)$

(No. of equations = No. of variables)

Unique solution: $X = A^{-1} b$

Least square solutions

<u>Case 2</u>: $(m < n)$ (under constrained problem)

(No. of equations < No. of variables)

In this case, there are infinitely many solutions. One way to get a meaningful solution is to formulate the following optimization problem:

Minimize $J = \|X\|_2, \quad \text{Subject to } AX = b$

Solution $X = A^{+}b, \quad where\ A^{+} = A^{T}\left(AA^{T}\right)^{-1} \quad (\text{right pseudo inverse})$

This solution WILL satisfy the equation $AX = b$ exactly.

Least square solutions

<u>Case 2</u>: $(m > n)$ (over constrained problem)

(No. of equations > No. of variables)

In this case, there is no solution. However, one way to get a meaningful (error minimizing) solution is to formulate the following optimization problem:

Minimize: $J = \|AX - b\|_2$

Solution: $X = A^+ b, \quad where\ A^+ = \left(A^T A\right)^{-1} A^T \quad (\text{left pseudo inverse})$

This solution **need not** satisfy the equation $AX = b$ exactly.

Generalized/Pseudo Inverse

- Left pseudo inverse: $A^{+} = \left(A^{T} A\right)^{-1} A^{T}$
- Right pseudo inverse: $A^{+} = A^{T}\left(AA^{T}\right)^{-1}$
- Properties:
  - (a) $A\,A^{+}A = A$
  - (b) $A^{+}A\,A^{+} = A^{+}$
  - (c) $\left(A\,A^{+}\right)^{T} = A\,A^{+}$
  - (d) $\left(A^{+}A\right)^{T} = A^{+}A$
  - (e) $A^{+} = A^{-1}$, if A is square and $|A| \neq 0$

Thanks for the Attention...!