Lecture – 9

Conversion Between State Space and Transfer Function Representations in Linear Systems – I

Dr. Radhakant Padhi

Asst. Professor

Dept. of Aerospace Engineering

Indian Institute of Science - Bangalore

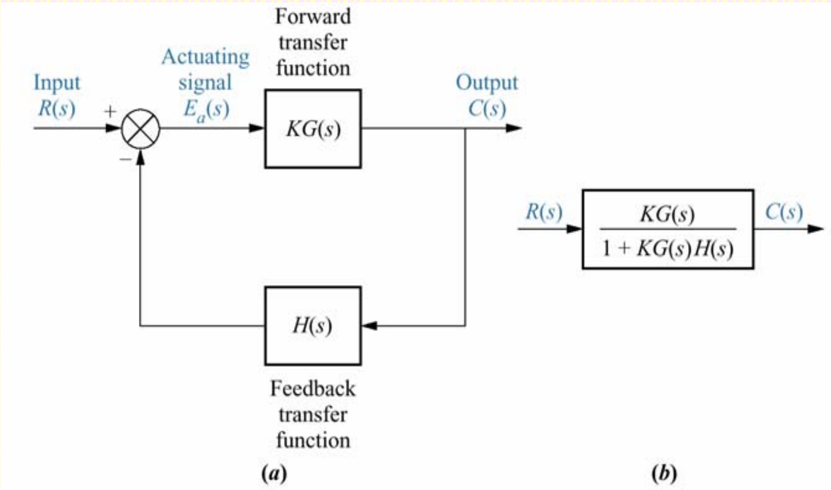
State Space Representation (noise free linear systems)

- **State Space form**

$$\dot{X} = AX + BU$$

$$Y = CX + DU$$

- **Transfer Function form**



A - System matrix- *n x n*

B - Input matrix- *n x m*

C - Output matrix- *p x n*

D - Feed forward matrix – *p x m*

Q: Is conversion between the two forms possible?

A: Yes.

Deriving Transfer Function Model From Linear State Space Model

- Known:

$$\dot{X} = AX + BU$$

$$Y = CX + DU$$

- Taking Laplace transform (with zero initial conditions)

$$sX(s) = AX(s) + BU(s)$$

$$Y(s) = CX(s) + DU(s)$$

Deriving Transfer functions from State Space Description

- The state equation can be placed in the form

$$(sI - A)X(s) = BU(s)$$

- Pre-multiplying both sides by $(sI - A)^{-1}$

$$X(s) = (sI - A)^{-1} BU(s)$$

- Substituting for *X (s)* in the output equation,

$$Y(s) = \underbrace{\left[C(sI - A)^{-1} B + D \right]}_{\text{Transfer Function Matrix } T(s)} U(s)$$

Example

- State space model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{D} u$$

- Using the expression for derived transfer function

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s+1)(s+2)}$$

Example: Detail Algebra

$$T(s) = C(sI - A)^{-1} B + D$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\frac{1}{s(s+3)+2} \right) \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s^2 + 3s + 2} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ s \end{bmatrix} = \frac{1}{s^2 + 3s + 2}$$

Deriving State Space Model From Transfer Function Model

- The process of converting transfer function to state space form is **NOT** unique. There are various "realizations" possible.
- All realizations are "equivalent" (i.e. properties do not change). However, one representation may have some advantages over others for a particular task.
- Possible representations:
  - First companion form (controllable canonical form)
  - Jordan canonical form
  - Alternate first companion form (Toeplitz first companion form)
  - Second companion form (observable canonical form)

First Companion Form: SISO Case
(Controllable canonical form)

$$H(s) = \frac{y(s)}{u(s)} = \left[\frac{1}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} \right]$$

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} \dot{y} + a_n y = u$$

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \quad \cdots \quad + a_n y = u$$

First Companion Form: SISO Case
(Controllable canonical form)

Choose output $y(t)$ and its $(n-1)$ derivatives as

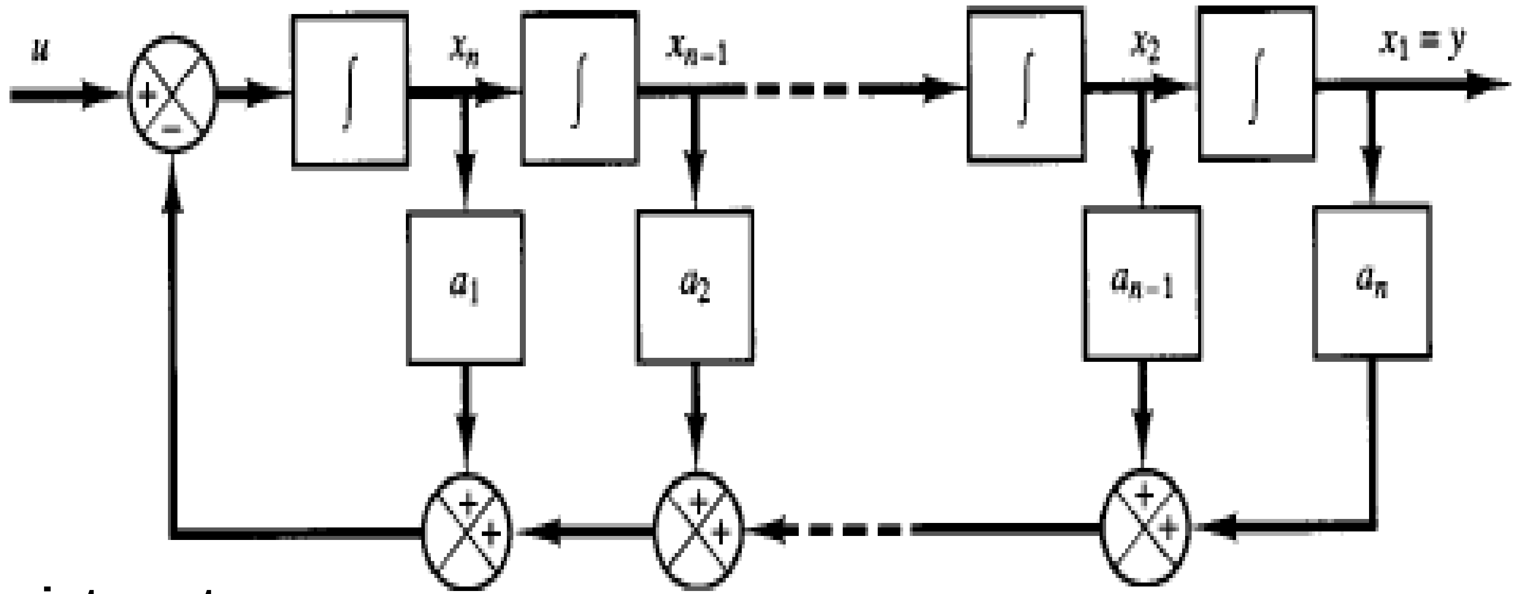
$$\begin{bmatrix} x_1 = y \\ x_2 = \dfrac{dy}{dt} \\ \vdots \\ x_n = \dfrac{d^{n-1}y}{dt^{n-1}} \end{bmatrix} \xrightarrow{\quad} \text{differentiating} \xrightarrow{\quad} \begin{bmatrix} \dot{x}_1 = \dfrac{dy}{dt} \\ \dot{x}_2 = \dfrac{d^2y}{dt^2} \\ \vdots \\ \dot{x}_n = \dfrac{d^n y}{dt^n} \end{bmatrix}$$

First Companion Form: SISO Case
(Controllable canonical form)

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix}
=
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-a_n & -a_{n-1} & -a_{n-2} & \cdots & \cdots & \cdots & \cdots & -a_1
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}
+
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u
$$

$$
y = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}
+ [0] u
$$

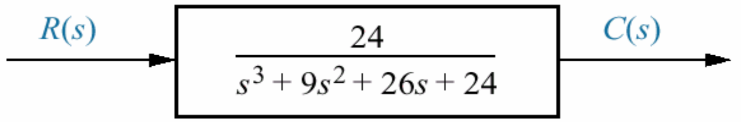
First Companion Form: SISO Case
(Controllable canonical form): Block Diagram

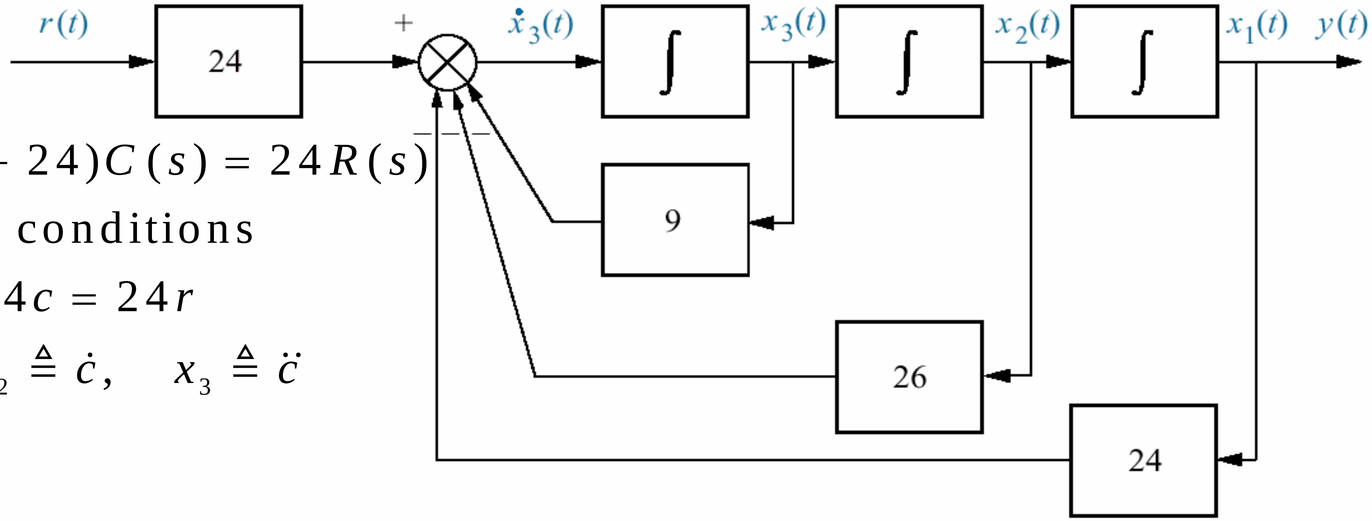


n integrators

Example:
(Constant in the Numerator)

3 integrators





$$(s^3 + 9s^2 + 26s + 24)C(s) = 24R(s)$$

with zero initial conditions

$$\dddot{c} + 9\ddot{c} + 26\dot{c} + 24c = 24r$$

Let $\quad x_1 \triangleq c, \quad x_2 \triangleq \dot{c}, \quad x_3 \triangleq \ddot{c}$

$$\dot{x}_1 = x_2$$

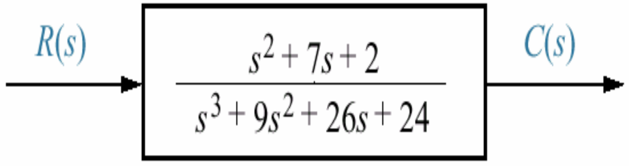
$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -24x_1 - 26x_2 - 9x_3 + 24r$$

$$y = c = x_1$$

Example:
(Polynomial in the Numerator)

For the block containing denominator

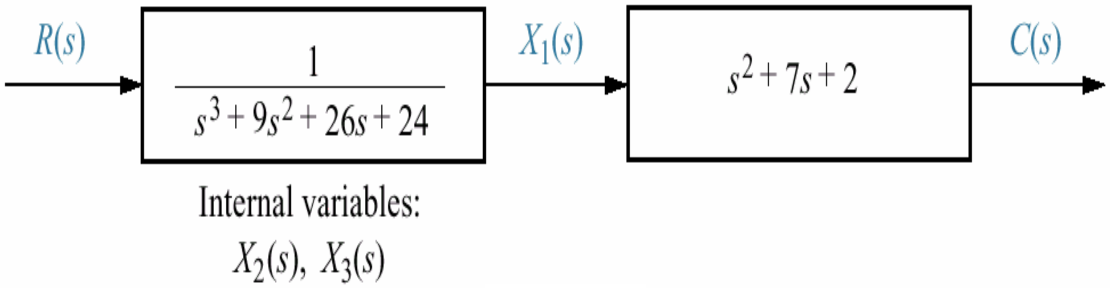


$$(s^3+9s^2+26s+24)X_1(s)=24R(s)$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -24x_1 - 26x_2 - 9x_3 + r$$



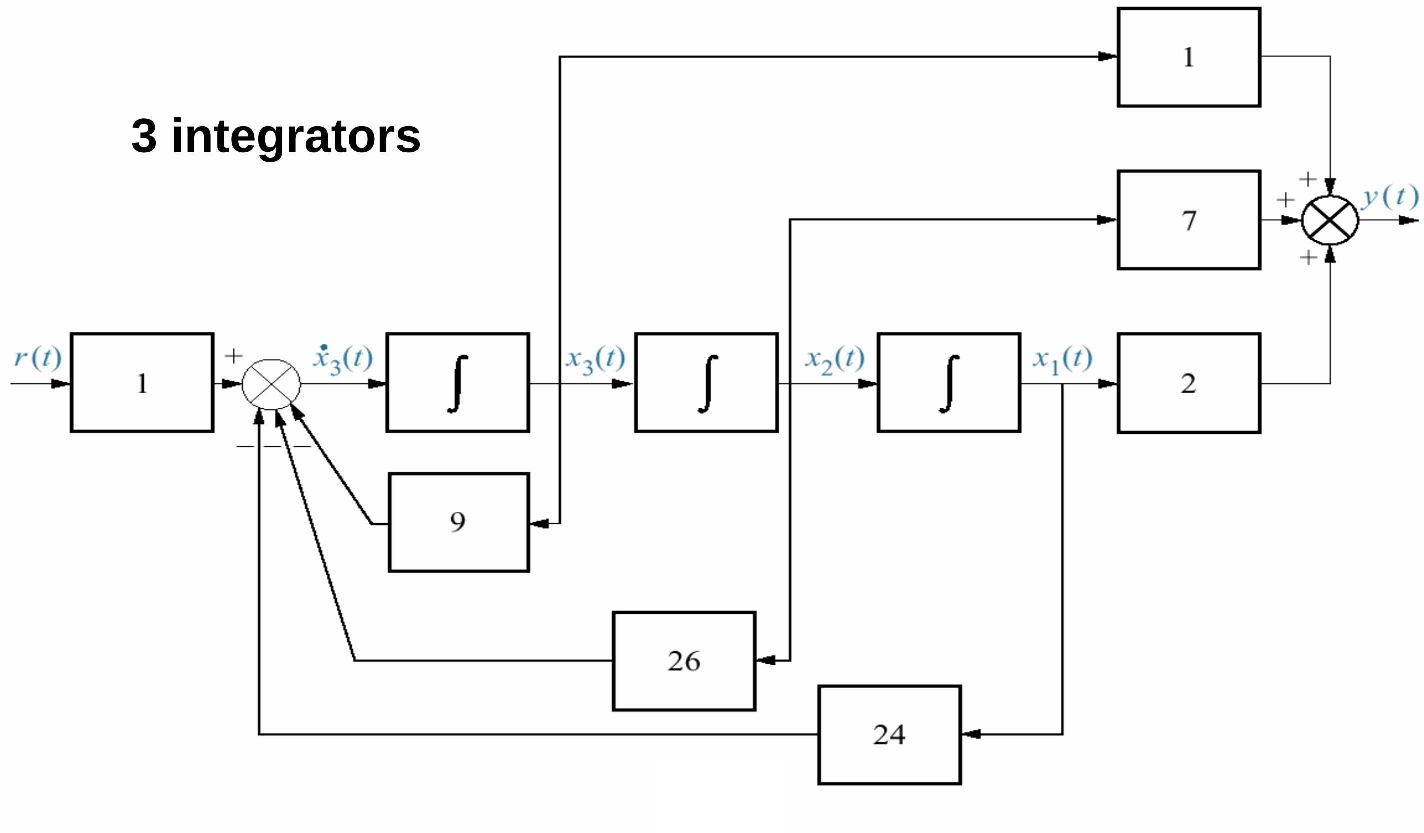
For Numerator Block

$$C(s)=(s^2+7s+2)X_1(s)$$

Taking inverse Laplace transform

$$y = x_3 + 7x_2 + 2x_1$$

Example With A Polynomial In The Numerator



Systems Having Single Input but Multiple Outputs

Generalization of the concepts discussed to this case is straight forward:

- *A* and *B* matrices remain same
- *C* and *D* matrices get modified

Example

$$\frac{y_1(s)}{u(s)} = H_1(s) = \frac{2s+3}{3s^2+4s+5} \quad \text{and}$$

$$\frac{y_2(s)}{u(s)} = H_2(s) = \frac{3s+2}{3s^2+4s+5}$$

$$H_1(s) = \frac{y_1(s)}{u(s)} = \left(\frac{z(s)}{u(s)}\right)\left(\frac{y_1(s)}{z(s)}\right) = \left(\frac{1}{3s^2+4s+5}\right)(2s+3)$$

Example – contd.

$$\ddot{z} + \frac{4}{3}\dot{z} + \frac{5}{3}z = \frac{1}{3}u$$

Define $x_1 \triangleq z, \; x_2 \triangleq \dot{z}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5/3 & -4/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/3 \end{bmatrix} u$$

$$y_1 = 2\dot{z} + 3z = 2x_2 + 3x_1$$

$$y_1 = \begin{bmatrix} 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u$$

Example – contd.

$$H_2(s) = \left(\frac{z(s)}{u(s)}\right)\left(\frac{y_2(s)}{z(s)}\right) = \left(\frac{1}{3s^2 + 4s + 5}\right)(3s + 2)$$

A and B are the same

$$y_2 = 3\dot{z} + 2z = 3x_2 + 2x_1$$

$$y_2 = \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u$$

Note: Block diagram representation is fairly straight forward.
The realization requires *n* integrators.

Jordan Canonical Form
(Non-repeated roots)

$$H(s) = \frac{y(s)}{u(s)} = d_0 + \frac{r_1}{s-\lambda_1} + \frac{r_2}{s-\lambda_2} + \quad \cdots \quad + \frac{r_n}{s-\lambda_n}$$

All the poles of the transfer function are distinct, i.e. no repeated poles

r_i's are called "residues" of the reduced transfer function $\left[H(s) - b_0\right]$

$$y(s) = d_0 u(s) + \frac{r_1 u(s)}{s-\lambda_1} + \frac{r_2 u(s)}{s-\lambda_2} + \quad \cdots \quad + \frac{r_n u(s)}{s-\lambda_n}$$

Let

$$\begin{bmatrix} x_1(s) = \dfrac{r_1 u(s)}{s-\lambda_1} \\ \vdots \\ x_n(s) = \dfrac{r_n u(s)}{s-\lambda_n} \end{bmatrix} \rightarrow \begin{bmatrix} \dot{x}_1 - \lambda_1 x_1 = r_1 u \\ \vdots \\ \vdots \\ \dot{x}_n - \lambda_n x_n = r_n u \end{bmatrix}$$

Jordan Canonical Form
(Non-repeated roots)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} d_0 \end{bmatrix} u$$

Jordan Canonical Form
(Non-repeated roots): Block Diagram

Jordan Canonical Form: Example
(Non-repeated roots)

Given
$$\frac{y(s)}{u(s)} = \frac{1}{(s+1)(s+2)}$$

By partial fraction,
$$y(s) = \left[\frac{1}{s+1} - \frac{1}{s+2}\right] u(s)$$

Define two transfer functions

$$x_1(s) = \frac{1}{s+1} u(s), \qquad x_2(s) = \frac{1}{s+2} u(s)$$

Jordan Canonical Form: Example
(Non-repeated roots)

This leads to

$$u(s) = x_1(s)(s+1), \quad u(s) = x_2(s)(s+2)$$

Differential equations corresponding to the x_1 , x_2

$$\dot{x}_1 = u - x_1$$
$$\dot{x}_2 = u - 2x_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{B} u$$

Output Equation

$$y = x_1 - x_2$$

$$y = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Jordan Canonical Form: Example

Repeated Roots

Following the same procedure

$$H(s) = \frac{y(s)}{u(s)} = \frac{2}{(s-3)^3}$$

Let

$$x_1(s) = \frac{x_2(s)}{s-3} \qquad \Rightarrow \qquad \dot{x}_1 - 3x_1 = x_2$$

$$x_2(s) = \frac{x_3(s)}{s-3} \qquad \Rightarrow \qquad \dot{x}_2 - 3x_2 = x_3$$

$$x_3(s) = \frac{u(s)}{s-3} \qquad \Rightarrow \qquad \dot{x}_3 - 3x_3 = u$$

Jordan Canonical Form: Example

Repeated Roots

- Output Equation

$$y(s)=\frac{2}{(s-3)^3}u(s)=2\frac{1}{(s-3)}\frac{1}{(s-3)}\underbrace{\left[\frac{u(s)}{(s-3)}\right]}_{x_3(s)}$$

$$=2\frac{1}{(s-3)}\frac{x_3(s)}{(s-3)}=2\frac{x_2(s)}{(s-3)}=2x_1(s)$$

- Finally

$$\underbrace{\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}}_{\dot{X}}=\underbrace{\begin{bmatrix}3&1&0\\0&3&1\\0&0&3\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}}_{X}+\underbrace{\begin{bmatrix}0\\0\\1\end{bmatrix}}_{B}u$$

$$y=\underbrace{\begin{bmatrix}2&0&0\end{bmatrix}}_{C}X+\underbrace{\begin{bmatrix}0\end{bmatrix}}_{D}u$$

Jordan Canonical Form:
What if complex conjugate roots?

Roots always exist in complex conjugate pairs!

$$H_{1,2} = \frac{\lambda + j\upsilon}{s + (\sigma - j\omega)} + \frac{\lambda - j\upsilon}{s + (\sigma + j\omega)}$$

$$= \frac{2\left[\lambda s + (\lambda\sigma - \omega\upsilon)\right]}{s^2 + 2\sigma s + (\sigma^2 + \omega^2)}$$

The system can be realized partially in other forms (like First companion form)

First companion form:

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(\sigma^2 + \omega^2) & -2\sigma \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y_{1,2} = \begin{bmatrix} 2(\lambda\sigma - \omega\upsilon) & 2\lambda \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

Thanks for the Attention….!