Lecture – 39

Integrator Back-stepping; Linear Quadratic (LQ) Observer

Dr. Radhakant Padhi

Asst. Professor

Dept. of Aerospace Engineering

Indian Institute of Science - Bangalore

Philosophy of Nonlinear Control Design Using Lyapunov Theory

Dr. Radhakant Padhi
Asst. Professor
Dept. of Aerospace Engineering
Indian Institute of Science - Bangalore

Philosophy of Feedback Control Design Using Lyapunov Theory

Motivation : $\dot{X} = f(X, U)$

Goal: Design $U = \varphi(X)$ such that

$\dot{X} = f(X, \varphi(X))$ is asymptotically stable

Design Idea:

* Choose a pdf $V_1(X)$
* Make $\dot{V}_1(X) \le -V_2(X)$, where $V_2(X) > 0$ (pdf)

Feedback Control Design Using Lyapunov Theory: An Example

Problem: Design a stabilizing controller for the following system

$$\dot{x} = ax^2 - x^3 + u$$

Solution: Let $V_1(X) = \frac{1}{2}x^2$

$$\dot{V}_1 = x\,\dot{x} = x\left(ax^2 - x^3 + u\right)$$

$$= ax^3 - x^4 + xu$$

Let us choose $V_2(X) = x^2$

$$\therefore \quad \dot{V}_1(X) \le -V_2(X)$$

$$\Rightarrow \quad ax^3 - x^4 + xu \le -x^2$$

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$$xu \le -x^2 + x^4 - ax^3$$

i.e $\boxed{u = -x + x^3 - ax^2}$

Analysis: $\dot{x} = ax^2 - x^3 - x + x^3 - ax^2$

$$\dot{x} = -x$$

Advantage: The closed loop system is globally asymptotically stable.

Problem: The benefitial nonlinearity got cancelled.

(which is not desirable)

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Let us choose:

$$V_2(X) = x^2 + x^4$$

Then

$$\dot{V}_1 \leq -V_2(X) \quad \text{leads to:}$$

$$ax^3 - x^4 + xu \leq -x^2 - x^4$$

$$ax^3 + xu \leq -x^2$$

$$ax^2 + u = -x \qquad \text{or} \qquad \boxed{u = -x - ax^2}$$

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Closed Loop system: $\dot{x} = ax^2 - x^3 - x - ax^2$

i.e. $\dot{x} = -x^3 - x$

$\Rightarrow$ The destabilizing nonlinearity got cancelled, but the benefitical nonlinearity is retained !

Another Problem: If $V_2(X) = x^2$,

$\dot{x} = -x$, only if a is accurate

If the actual parameter value is $\bar{a}$, then the feedback loop operates with

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$$\dot{x} = -x + \underbrace{\left(\bar{a} - a\right) x^2}_{\text{Can be potentially destabilizing term if } (\bar{a}-a) \text{ is high}}$$

i.e. The global stability reduces to local stability.

This excits robustness issues!

However, if $V_2\left(X\right)$ is made "Sufficiently powerful",

then the destabilizing effect can be minimized.

Hence, Lyapunov based designs can be "very robust"

Control Design Using Integrator Back-stepping

Dr. Radhakant Padhi

Asst. Professor

Dept. of Aerospace Engineering

Indian Institute of Science - Bangalore

Integrator Back-stepping

Problem : Design a state feedback asymptotically stabilizing controller for the following system

$$\boxed{\begin{aligned} \dot{X} &= f(X) + g(X)\xi \\ \dot{\xi} &= u \end{aligned}}$$

where $X \in \mathbb{R}^n,\ \xi \in \mathbb{R},\ u \in \mathbb{R}$

Note: $\begin{bmatrix} X \\ \xi \end{bmatrix} \in \mathbb{R}^{n+1}$: State of the system, u : Control input (single input)

Integrator Back-stepping

Assumptions:

- $f, g : D \to \mathbb{R}^n$ are smooth
- $f(0) = 0$
- Considering state ξ as a "control input" of subsystem (1)

  we assume that $\exists$ a state feedback control law

  of the from $\xi = \varphi(X), \varphi(0) = 0$. Moreover,

  $\exists$ a Lyapunov function $V_1 : D \to \mathbb{R}^+$ such that

$$\dot{V}_1(X) = \left(\frac{\partial V_1}{\partial X}\right)^T \left[f(X) + g(X)\varphi(X) \right] \le -V_a(X), \ \forall X \in D$$

where, $V_a(X) : D \to \mathbb{R}^+$ is a pdf function.

Integrator Back-stepping

An important observation:

When $X = 0, \ \xi = \varphi(0) = 0$ & $\dot{X} = f(0) = 0$

(i.e. everything is nice)

However, when $\xi \to 0, \ \dot{X} = f(X)$ and hence $X \not\searrow 0$

in general. That is the core problem!

$\therefore$ We need some algebraic manipulation as follows.

Integrator Back-stepping

Step - 1:

$$\dot{X} = f(X) + g(X)\xi + g(X)\varphi(X) - g(X)\varphi(X)$$

$$= f(X) + g(X)\varphi(X) + g(X)\underbrace{\left[\xi - \varphi(X)\right]}_{z}$$

$$= f(X) + g(X)\varphi(X) + g(X)z$$

By this construction, when $z \to 0$, $\dot{X} = f(X) + g(X)\varphi(X)$

which is asymptotically stable (i.e. $X \to 0$)!

Integrator Back-stepping

$$\dot{z} = \dot{\xi} - \dot{\varphi}$$

$$= \underbrace{u - \dot{\varphi}}_{v} \qquad \left[\begin{array}{l}\text{This is backstepping, since } \varphi(X) \text{ is} \\ \text{stepped back by differentiation}\end{array}\right]$$

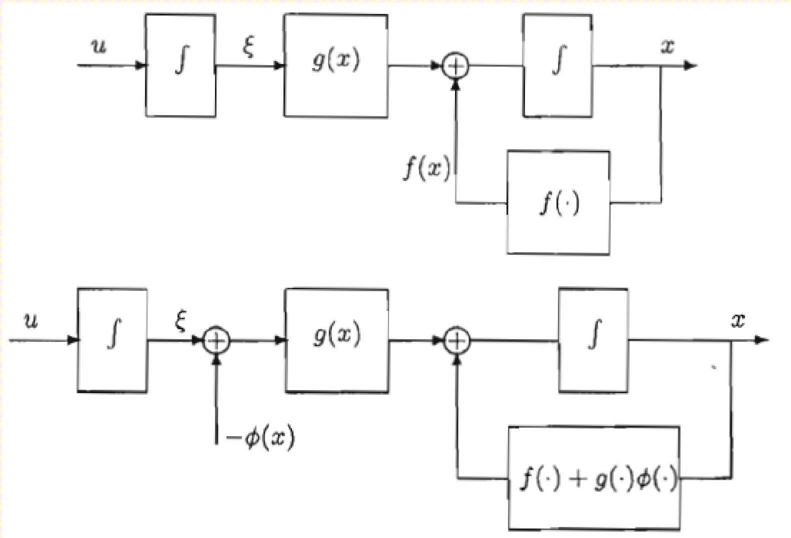
So, we have

$$\boxed{\begin{aligned} \dot{X} &= f(X) + g(X)\varphi(X) + g(X)z \\ \dot{z} &= v \end{aligned}}$$

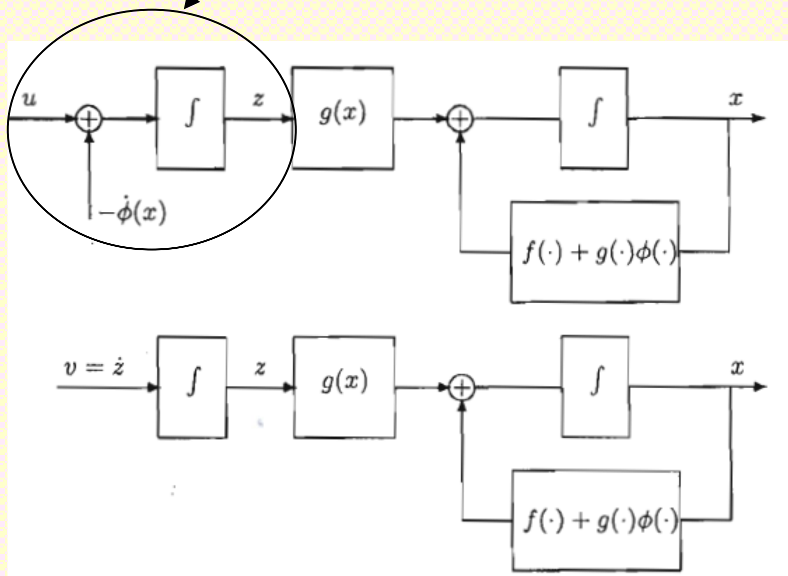
[This system is equivalent to the original system]

<u>Note</u>: $\dot{\varphi} = \left(\dfrac{\partial \varphi}{\partial X}\right)^T \dot{X} = \left(\dfrac{\partial \varphi}{\partial X}\right)^T \left[f(X) + g(X)\xi \right]$

Back-stepping: Conceptual Block Diagram



Back-stepping



Ref: H. J. Marquez, Nonlinear Control Systems: Analysis and Design, Wiley, 2003.

Integrator Back-stepping

Step-2: Let $V(X,z) = V_1(X) + \frac{1}{2} z^2$

Then
$$\dot{V} = \underbrace{\left(\frac{\partial V_1}{\partial X}\right)^T \left[f(X) + g(X)\varphi(X)\right.}_{\le -V_a(X)} \left. + g(X) z \right] + z\,v$$

$$\le -V_a(X) + \left[\left(\frac{\partial V_1}{\partial X}\right)^T g(X) + v \right] z$$

Let $v = -\left(\frac{\partial V_1}{\partial X}\right)^T g(X) - k\,z, \quad k > 0$

Then $\dot{V} \le \underbrace{-V_a(X)}_{\text{ndf}} \underbrace{-k\,z^2}_{nddf} < 0$ (ndf)

Integrator Back-stepping

Control Solution:

$$v = u - \dot{\varphi} = -\left(\frac{\partial V_1}{\partial X}\right)^T g(X) - k\,z$$

$$u = \dot{\varphi} - \left(\frac{\partial V_1}{\partial X}\right)^T g(X) - k\left[\xi - \varphi(X)\right]$$

$$\text{where,}\quad \dot{\varphi} = \left(\frac{\partial \varphi}{\partial X}\right)^T \left[f(X) + g(X)\,\xi\right], k > 0$$

[Note: In the design, there is a need to design $\varphi(X)$ first]

Integrator Back-stepping: An Example

Problem:

$$\dot{x}_1 = a x_1^2 - x_1^3 + x_2$$

$$\dot{x}_2 = u$$

Solution :

$$X = x_1,\ f(x_1) = a x_1^2 - x_1^3\ ,\ \xi = x_2\ ,\ g(x_1) = 1$$

To find $\varphi(x_1)$:

$$V_1(x_1) = \frac{1}{2} x_1^2$$

$$\dot{V}_1 = x_1 \dot{x}_1 = x_1 \left(a x_1^2 - x_1^3 + x_2 \right) \le -\underbrace{V_a(x_1)}_{x_1^2 + x_1^4}$$

Integrator Back-stepping: An Example

$$a x_1^3 - \cancel{x_1^4} + x_1 x_2 \le -x_1^2 - \cancel{x_1^4}$$

$$x_1\left(a x_1^2 + x_2\right) \le -x_1^2$$

Let $\quad a x_1^2 + x_2 = -x_1$

$$\Rightarrow \quad x_2 = \left(-a x_1^2 - x_1\right) \triangleq \varphi(x_1)$$

Modified system : $\dot{x}_1 = a x_1^2 - x_1^3 + \varphi(x_1) + \overbrace{\left[x_2 - \varphi(x_1)\right]}^{z}$

$$\dot{z} = v \triangleq \left(\cancel{\dot{x}_2}^{\,u} - \dot{\varphi}(x_1)\right)$$

Let $\quad V(x_1, z) = V_1(x_1) + \dfrac{1}{2} z^2$

$$\dot{V} = \dot{V}_1 + z\,v = \left(\frac{\partial V_1}{\partial x_1}\right)\left[a x_1^2 - x_1^3 + \varphi(x_1)\right] + \left(\frac{\partial V_1}{\partial x_1} + v\right) z$$

Integrator Back-stepping: An Example

Let $v = -\left(\frac{\partial V_1}{\partial x_1}\right) - k\,z\ ,\ k > 0$

$$u - \dot{\varphi} = -x_1 - k\left[x_2 - \varphi(x_1)\right]$$

$$u = \frac{\partial \varphi}{\partial x_1}\left(a x_1^{\,2} - x_1^{\,3} + x_2\right) - x_1 - k\left[x_2 - \left(-a x_1^{\,2} - x_1\right)\right]$$

$$= \left(-2a x_1 - 1\right)\left(a x_1^{\,2} - x_1^{\,3} + x_2\right) - x_1 - k\left[x_2 + a x_1^{\,2} + x_1\right]$$

$$u = -\left(1 + 2a x_1\right)\left(a x_1^{\,2} - x_1^{\,3} + x_2\right) - x_1 - k\left(x_1 + x_2 + a x_1^{\,2}\right)$$

where $k > 0$

Integrator Back-stepping: An Example

Note: The composite Lyapunov function is:

$$
\begin{aligned}
V &= V_1 + \frac{1}{2} z^2 \\
&= \frac{1}{2} x_1^{\;2} + \frac{1}{2}\left[x_2 - \varphi(x_1) \right]^2 \\
&= \frac{1}{2} x_1^{\;2} + \frac{1}{2}\left(x_2 + x_1 + a x_1^{\;2} \right)^2
\end{aligned}
$$

Integrator Back-stepping: More General Case

System Dynamics:

$$\dot{X} = f(X) + g(X)\xi_1$$

$$\dot{\xi}_1 = \xi_2$$

$$\dot{\xi}_2 = u$$

Idea : Successive iteration.

[Note: The procedure for n^{th} order system is entirely analogous]

Integrator Back-stepping: More General Case

Step-1: Consider the subsystem

$$\dot{X} = f(X) + g(X)\xi_1$$
$$\dot{\xi}_1 = \xi_2$$

Assumption:

$\xi_1 = \varphi(X)$ is a stabiliting feedback law for

$$\dot{X} = f(X) + g(X)\xi_1$$

and $V_1(X)$ is the coresponding Lyapunov function.

Integrator Back-stepping: More General Case

By the result obtained before, we have

$$\xi_2 = \left(\frac{\partial \varphi(X)}{\partial X}\right)^T \underbrace{\left[f(X) + g(X)\xi_1\right]}_{\dot{X}} - \left(\frac{\partial V_1}{\partial X}\right)^T g(X) - k\left[\xi_1 - \varphi(X)\right]$$

$$\triangleq \varphi_1(X, \xi_1) \qquad\qquad (k > 0)$$

We also have $V_2 = V_1 + \frac{1}{2}\left[\xi_1 - \varphi(X)\right]^2$

Integrator Back-stepping: More General Case

Step - 2:

$$\dot{X}_1 \triangleq \begin{bmatrix} \dot{X} \\ \dot{\xi}_1 \end{bmatrix} = \underbrace{\begin{bmatrix} f(X) + g(X)\xi_1 \\ 0 \end{bmatrix}}_{f_1(X_1)} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{g_1(X_1)} \xi_2$$

$$\dot{\xi}_2 = u \qquad \text{where,} \quad X_1 \triangleq \begin{bmatrix} X \\ \hline \xi_1 \end{bmatrix}$$

$\therefore$ Using the same idea,

$$u = \left(\frac{\partial \varphi_1}{\partial X_1}\right)^T \left[f_1(X_1) + g_1(X_1)\xi_2 \right] - \left(\frac{\partial V_2}{\partial X_1}\right)^T g_1(X_1) - k_1\left[\xi_2 - \varphi_1(X_1)\right], \quad k_1 > 0$$

and $V = V_2 + \frac{1}{2}\left[\xi_2 - \varphi_1(X_1)\right]^2 = V_1 + \frac{1}{2}\left[\xi_1 - \varphi(X)\right]^2 + \frac{1}{2}\left[\xi_2 - \varphi_1(X_1)\right]^2$

Integrator Back-stepping for Strict Feedback Systems

System Dynamics

$$\dot{X} = f(X) + g(X)\xi_1$$

$$\dot{\xi}_1 = f_1(X,\xi_1) + g_1(X,\xi_1)\xi_2$$

$$\dot{\xi}_2 = f_2(X,\xi_{1,}\xi_2) + g_2(X,\xi_1,\xi_2)\xi_3$$

$$\vdots$$

$$\vdots$$

$$\dot{\xi}_k = f_k(X,\xi_1,\ldots,\xi_k) + g_k(X,\xi_1,\ldots,\xi_k)u$$

Strong Assumption:

$$g_1(X,\xi_1), g_2(X,\xi_1,\xi_2),\ldots, g_k(X,\xi_1,\ldots,\xi_k) \neq 0$$

over the domain of interest $\forall t$

Integrator Back-stepping for Strict Feedback Systems

Special Case: $\dot{X} = f(X) + g(X)\xi$

$$\dot{\xi} = f_a(X,\xi) + g_a(X,\xi)u$$

Solution:

Define $\dot{\xi} = v$ and carryout the design for v as before.

Finally $f_a(X,\xi) + g_a(X,\xi)u = v$

i.e. $$\boxed{u = \frac{1}{g_a(X,\xi)}\left[v - f_a(X,\xi)\right]}$$

Note: By assumption, $g_a(X,\xi) \neq 0 \;\; \forall\, t$

Linear Quadratic (LQ) Observer

Dr. Radhakant Padhi
Asst. Professor
Dept. of Aerospace Engineering
Indian Institute of Science - Bangalore

Why Observers?

- State feedback control designs need the state information for control computation
- In practice all the state variables are not available for feedback. Possible reasons are:
  - Non-availability of sensors
  - Expensive sensors
  - Quality of some sensors may not acceptable due to noise (its an issue in output feedback control design as well)
- A state observer estimates the state variables based on the measurement of some of the output variables as well as the plant information.

Observer

- An observer is a dynamic system whose output is an estimate of the state vector X

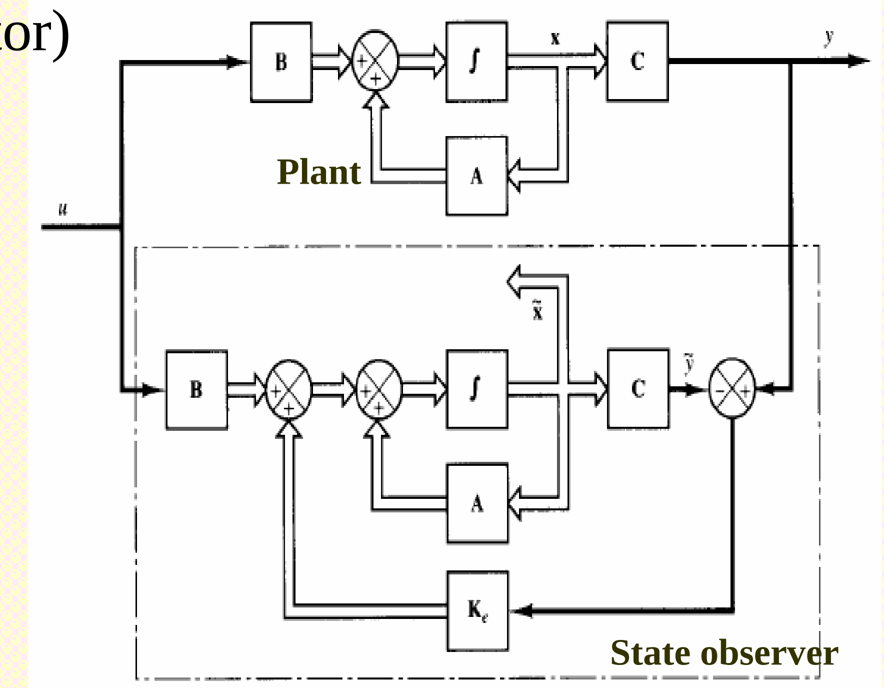
    - Full-order Observer
    - Reduced-order Observer

- **Observability** condition must be satisfied for designing an observer (this is true for filter design as well)

Observer Design for Linear Systems

Plant : $\dot{X} = AX + BU$

$Y = CX$ (sensor output vector)



Let the observed state be $\hat{X}$ and the **Observer dynamics** be

$$\dot{\hat{X}} = \hat{A}\hat{X} + \hat{B}U + K_e Y$$

Error : $\tilde{X} \triangleq \left(X - \hat{X} \right)$

Observer Design for Linear Systems

Error Dynamics: $\dot{\tilde{X}} = \dot{X} - \dot{\hat{X}}$

$$= \left(AX + BU \right) - \left(\hat{A}\hat{X} + \hat{B}U + K_e Y \right)$$

Add and Substract $\tilde{A}X$ and substitute $Y = CX$

$$\dot{\tilde{X}} = AX - \hat{A}X + \hat{A}X - \hat{A}\hat{X} + BU - \hat{B}U - K_e C X$$

$$= (A - \hat{A})X + \hat{A}(X - \hat{X}) + (B - \hat{B})U - K_e C X$$

$$= \hat{A}\tilde{X} + \left(A - \hat{A} - K_e C \right) X + \left(B - \hat{B} \right) U$$

Goals: 1. Make the error dynamics independent of X

($\because$ X may be large, even though $\tilde{X}$ may be small)

2. Eliminate the effect of U from eror dynamics

Observer Design for Linear Systems

This can be done by enforcing $A - \hat{A} - K_e C = 0$

and $B - \hat{B} = 0$

Necessary and sufficient condition for the existence of $\boldsymbol{K_e}$:

The system should be "observable".

This results in

$$\hat{A} = A - K_e C$$

$$\hat{B} = B$$

Observer dynamics: $$\dot{\hat{X}} = A\hat{X} + BU + K_e\left(Y - C\hat{X}\right)$$

Observer Design: Full Order

- Order of the observer is same as that of the system (i.e. all states are estimated, irrespective of whether they are measured or not).
- Goal: Obtain gain K_e such that the error dynamics are asymptotically stable with sufficient speed of response.

  This means that $\hat{A} = A - K_e C$ is Hurwitz (i.e. it has all eigenvalues strictly in the left half plane.
- Note: $\hat{A}^T = A^T - C^T K_e^T$ and the eigen values of both $\hat{A}$ and $\hat{A}^T$ are ***same!***

Comparison of Control and Observer Design Philosophies

Control Design

- CL Dynamics

$$\dot{X} = (A - BK)X$$

- Objective

$$X(t) \to 0, \quad \text{as} \quad t \to \infty$$

Observer Design

- CL Error Dynamics

$$\dot{\tilde{X}} = \hat{A}\tilde{X} = (A - K_e C)\tilde{X}$$

- Objective

$$\tilde{X}(t) \to 0, \quad \text{as} \quad t \to \infty$$

- Notice that

$$\lambda(A - K_e C) = \lambda\left[(A - K_e C)^T\right]$$
$$= \lambda\left(A^T - C^T K_e^T\right)$$

Algebraic Riccati Equation (ARE) Based Observer Design

System

$$\dot{X} = AX + BU$$
$$Y = CX$$

$$M = \left[B \middle| AB \middle| \cdots \middle| A^{n-1} B \right]$$
$$N = \left[C^T \middle| A^T C^T \middle| \cdots \middle| {A^T}^{n-1} C^T \right]$$

LQR Design

$$U = -K X$$

Dual System

$$\dot{Z} = A^T Z + C^T V$$
$$n = B^T Z$$

$$M = \left[C^T \middle| A^T C^T \middle| \cdots \middle| {A^T}^{n-1} C^T \right]$$
$$N = \left[B \middle| AB \middle| \cdots \middle| A^{n-1} B \right]$$

ARE Based Observer Design

CL system (control design)

$$\dot{X} = (A - BK)X$$

$$X \to 0 \quad as \quad t \to \infty$$

$$K = R^{-1}B^T P, \quad P > 0$$

where,

$$PA + A^T P - PBR^{-1}B^T P + Q = 0$$

Error Dynamics

$$\dot{\tilde{X}} = (A - K_e C)\tilde{X}$$

$$(A - K_e C)^T = A^T - C^T K_e^T$$

Acts like a controller gain

Analogous

$$K_e^T = R^{-1}CP$$

where,

$$PA^T + AP - PC^T R^{-1} CP + Q = 0$$

Observer Dynamics

$$\dot{\hat{X}} = A\hat{X} + BU + K_e(Y - C\tilde{X})$$

Continuous-time Kalman Filter Design for Linear Time Invariant (LTI) Systems

Dr. Radhakant Padhi
Asst. Professor
Dept. of Aerospace Engineering
Indian Institute of Science - Bangalore

Problem Statement

System Dynamics: $\dot{X} = AX + BU + GW$ $\quad W(t)$: Process noise vector

Measured Output: $Y = CX + V$ $\quad V(t)$: Sensor noise vector

Assumptions:

(i) $\quad X(0) \sim \left(\tilde{X}_0, P_0\right), \quad W(t) \sim \left(0, Q\right)$ and $V(t) \sim \left(0, R\right)$

are "mutually orthogonal" $\left[X(0): \text{ initial condition for } X\right]$

(ii) $\quad W(t)$ and $V(t)$ are uncorrelated white noise

(iii) $\quad E\left[W(t)\, W^T(t+\tau)\right] = Q\,\delta(\tau), \quad Q \geq 0 \quad$ (psdf)

$$E\left[V(t)\, V^T(t+\tau)\right] = R\,\delta(\tau), \quad R > 0 \quad \text{(pdf)}$$

Problem Statement

Objective:

To obtain an estimate of the state vector $\hat{X}(t)$ using the state dynamics as well as a "sequence of measurements" as accurate as possible.

i.e., to make sure that the error $\tilde{X}(t) \triangleq \left[X(t) - \hat{X}(t) \right]$ becomes very small $\left(\text{ideally } \tilde{X}(t) \to 0 \right)$ as $t \to \infty$.

Observer/Estimator/Filter Dynamics

$$\dot{\hat{X}} = A\hat{X} + BU + K_e\left(Y - \hat{Y}\right)$$

where $(i)\ \hat{X} = E(X)$: Estimate of the state X

$(ii)\ \hat{Y} = E(Y)$: Estimate of the output Y

$$= E\left(CX + V\right)$$

$$= E\left(CX\right) + E\left(V\right)$$

$$= CE\left(X\right) \quad \left(\because E(V) = 0\right)$$

$$= C\hat{X}$$

$(iii)\ K_e$: Estimator/Filter/Kalman Gain

Problem : How to design K_e ?

Solution: Summary

(i) Initialize $\hat{X}(0)$

(ii) Solve for Riccati matrix P from the Filter ARE:

$$AP + PA^T - PC^T R^{-1} CP + GQG^T = 0$$

(iii) Compute Kalman Gain:

$$K_e = PC^T R^{-1}$$

(iv) Propagate the Filter dynamics:

$$\dot{\hat{X}} = A\hat{X} + BU + K_e\left(Y - C\hat{X}\right)$$

where Y is the measurement vector

Thanks for the Attention...!