Lecture – 24

Calculus of Variations : An Overview

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Fundamental Theorems of Calculus

<u>Theorem – 1</u>: $$\frac{d}{dx}\int_a^x f(\sigma)\,d\sigma = f(x), \text{ provided } f(x) \text{ is continuous}$$

<u>Theorem – 2</u>: $$\frac{d}{dx}\int_a^b f(x,y)\,dy = \int_a^b \frac{\partial f(x,y)}{\partial x}\,dy$$

provided $f(x,y)$ has continuous partial derivative $(\partial f/\partial x)$

<u>Theorem – 3</u>:

$$\frac{d}{dx}\int_{\psi_1(x)}^{\psi_2(x)} f(x,y)\,dy = \int_{\psi_1(x)}^{\psi_2(x)} \frac{\partial f(x,y)}{\partial x}\,dy + \left[\frac{d\psi_2}{dx} f\big(x,\psi_2(x)\big) - \frac{d\psi_1}{dx} f\big(x,\psi_1(x)\big)\right]$$

provided f, ψ_1, ψ_2 have continuous partial derivatives with respect to x

Calculus of Variations: Basic Concepts

- **Function** (to each value of the independent variable, there is a corresponding value of the dependent variable)

$$x(t) = 2t^3 + 3t$$

- **Functional** (to each function, there is a corresponding value of the dependent variable)

$$J\left(x(t)\right) = \int_0^1 x(t)\,dt$$

$$= \int_0^1 \left(2t^3 + 3t\right) dt = 2$$

- Increment of a function

$$\Delta x \triangleq x(t + \Delta t) - x(t)$$

- Increment of a functional

$$\Delta J \triangleq J\left(x(t) + \delta x(t)\right) - J\left(x(t)\right)$$

Example

$$J = \int_{t_0}^{t_f} \left[2x^2(t) + 1 \right] dt$$

$$\Delta J = J\left(x(t) + \delta x(t) \right) - J\left(x(t) \right)$$

$$= \int_{t_0}^{t_f} \left[2\left(x(t) + \delta x(t) \right)^2 + 1 \right] dt - \int_{t_0}^{t_f} \left[2x^2(t) + 1 \right] dt$$

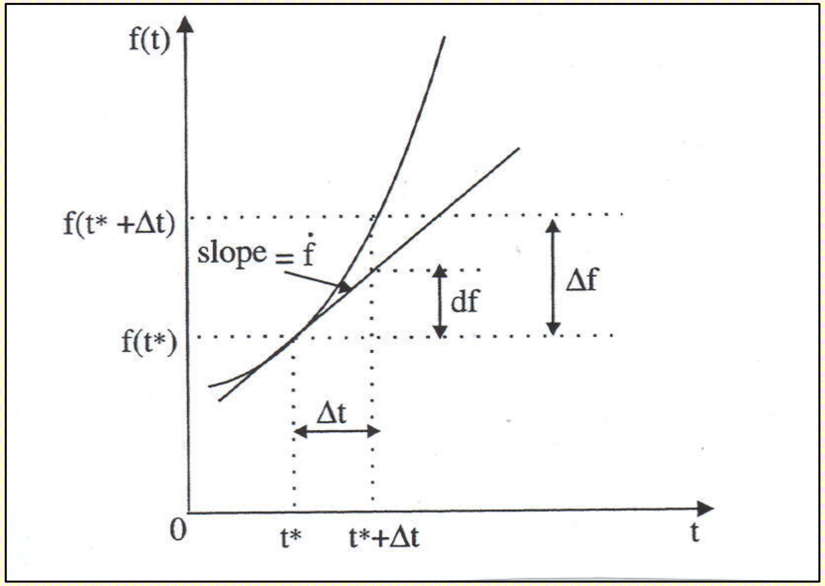
$$= \int_{t_0}^{t_f} \left(\left[2\left(x(t) + \delta x(t) \right)^2 + 1 \right] - \left[2x^2(t) + 1 \right] \right) dt$$

$$= \int_{t_0}^{t_f} \left(\left[2\left(x^2(t) + 2x(t)\delta x(t) + \left(\delta x(t) \right)^2 \right) + 1 \right] - \left[2x^2(t) + 1 \right] \right) dt$$

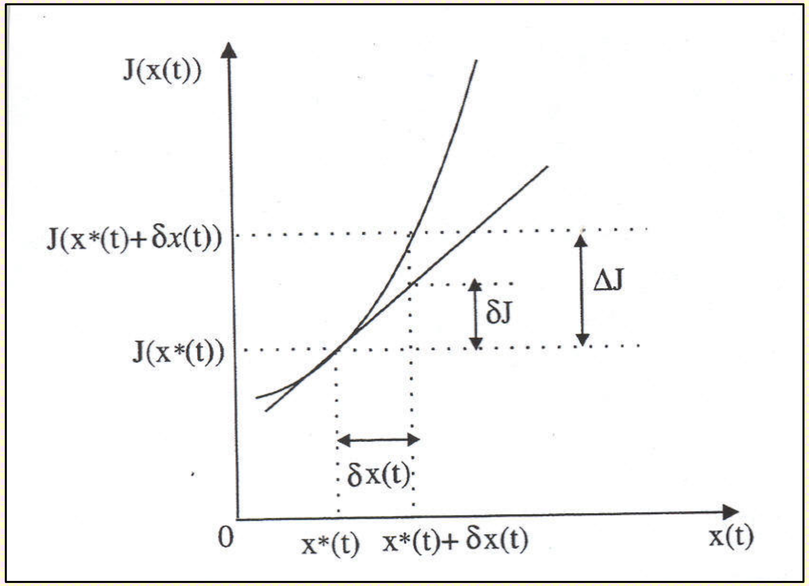
$$= \int_{t_0}^{t_f} \left[4x(t)\delta x(t) + 2\left(\delta x(t) \right)^2 \right] dt$$

Calculus of Variations: Basic Concepts

Function and its increment



Functional and its increment



Reference: D. S. Naidu, *Optimal Control Systems*, CRC Press, 2002.

Calculus of Variations: Basic Concepts

Differential of a function

$$\Delta f^* = f\left(t^* + \Delta t\right) - f\left(t^*\right)$$

$$= \underbrace{\left(\left.\frac{df}{dt}\right|_{t^*}\right)\Delta t}_{df\text{: First deviation}} + \underbrace{\frac{1}{2!}\left(\left.\frac{d^2 f}{dt^2}\right|_{t^*}\right)(\Delta t)^2}_{d^2 f\text{: Second deviation}} + \cdots$$

$$= df + d^2 f + \cdots$$

$$df^* = \lim_{\Delta t \to 0} \Delta f^* = \lim_{\Delta t \to 0}\left(\left.\frac{df}{dt}\right|_{t^*}\right)\Delta t$$

$$i.e.\ df = \left(\dot{f}\right)\Delta t \quad \text{in general}$$

Variation of a functional

$$\Delta J = J\left(x(t) + \delta x(t)\right) - J\left(x(t)\right)$$

$$= \underbrace{\left(\frac{\partial J}{\partial x}\right)\delta x}_{\delta J\text{: First variation}} + \underbrace{\frac{1}{2!}\left(\frac{\partial^2 J}{\partial x^2}\right)(\delta x)^2}_{\delta^2 J\text{: Second variation}} + \cdots$$

$$= \delta J + \delta^2 J + \cdots$$

$$\delta J = \left(\frac{\partial J}{\partial x}\right)\delta x$$

Calculus of Variations: Basic Concepts

<u>Result – 1</u>:
Derivative of variation = Variation of derivative

$$\frac{d}{dt}\left[\delta x(t)\right] = \frac{d}{dt}\left[x(t)-x^*(t)\right]$$

$$= \frac{dx(t)}{dt} - \frac{dx^*(t)}{dt}$$

$$= \delta\left[\dot{x}(t)\right]$$

<u>Result – 2</u>:
Integration of variation = Variation of integration

$$\int_{t_0}^{t} \delta x(\tau)\, d\tau = \int_{t_0}^{t}\left[x(\tau)-x^*(\tau)\right] d\tau$$

$$= \int_{t_0}^{t} x(\tau)\, d\tau - \int_{t_0}^{t} x^*(\tau)\, d\tau$$

$$= \delta\left[\int_{t_0}^{t} x(\tau)\, d\tau\right]$$

Exercise

Evaluate the variation of :

$$J\left(x(t)\right) = \int_{t_0}^{t_f} \left[2x^2(t) + 3x(t) + 4 \right] dt$$

<u>Note:</u>

By 'variation', we mean 'first variation' (by default)

Solution

Method - 1:

$$\Delta J = J\big(x(t)+\delta x(t)\big) - J\big(x(t)\big)$$

$$= \int_{t_0}^{t_f} \left[2(x(t)+\delta x(t))^2 + 3(x(t)+\delta x(t)) + 4 \right] dt - \int_{t_0}^{t_f} \left[2x^2(t) + 3x(t) + 4 \right] dt$$

$$= \int_{t_0}^{t_f} \left[2\left[x^2 + 2x\,\delta x + (\delta x)^2 \right] + 3(x+\delta x) + 4 - \left[2x^2 + 3x + 4 \right] \right] dt$$

$$= \int_{t_0}^{t_f} \left[4x\,\delta x + 2(\delta x)^2 + 3\,\delta x \right] dt$$

$$= \int_{t_0}^{t_f} [4x+3]\,\delta x\, dt \qquad \big(\text{Neglecting the higher order term}\big)$$

Solution

<u>Method - 2 :</u>

$$\delta J = \left(\frac{\partial J}{\partial x} \right) \delta x$$

$$= \frac{\partial}{\partial x} \left[\int_{t_0}^{t_f} \left[2x^2(t) + 3x(t) + 4 \right] dt \right] \delta x(t)$$

$$= \left[\int_{t_0}^{t_f} \frac{\partial}{\partial x} \left[2x^2 + 3x + 4 \right] \delta x \right] dt$$

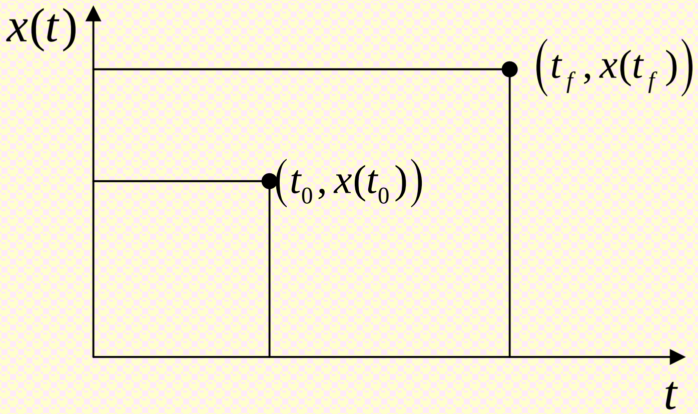
(Variation of integral = Integral of variation)

$$= \int_{t_0}^{t_f} \left[4x + 3 \right] \delta x \, dt$$

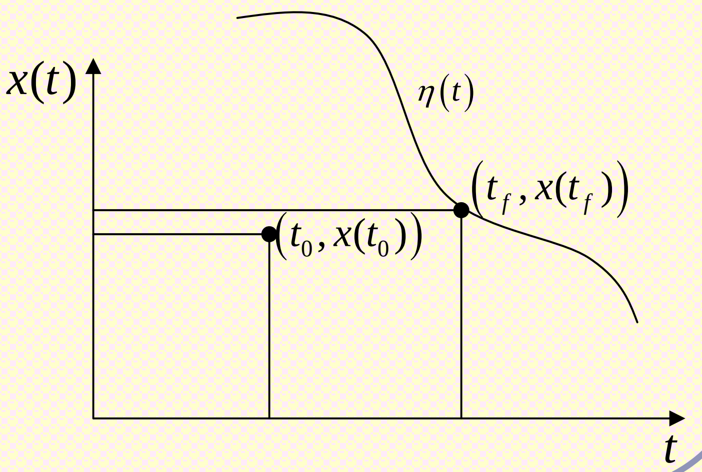
Boundary Conditions

- Fixed End Point Problems

$$\left(t_0, x(t_0)\right)\text{: Specified}$$
$$\left(t_f, x(t_f)\right)\text{: Specified}$$



- Free End Point Problems
  - Completely free
  - May be required to lie on a curve $\eta(t)$



Optimum of a Functional

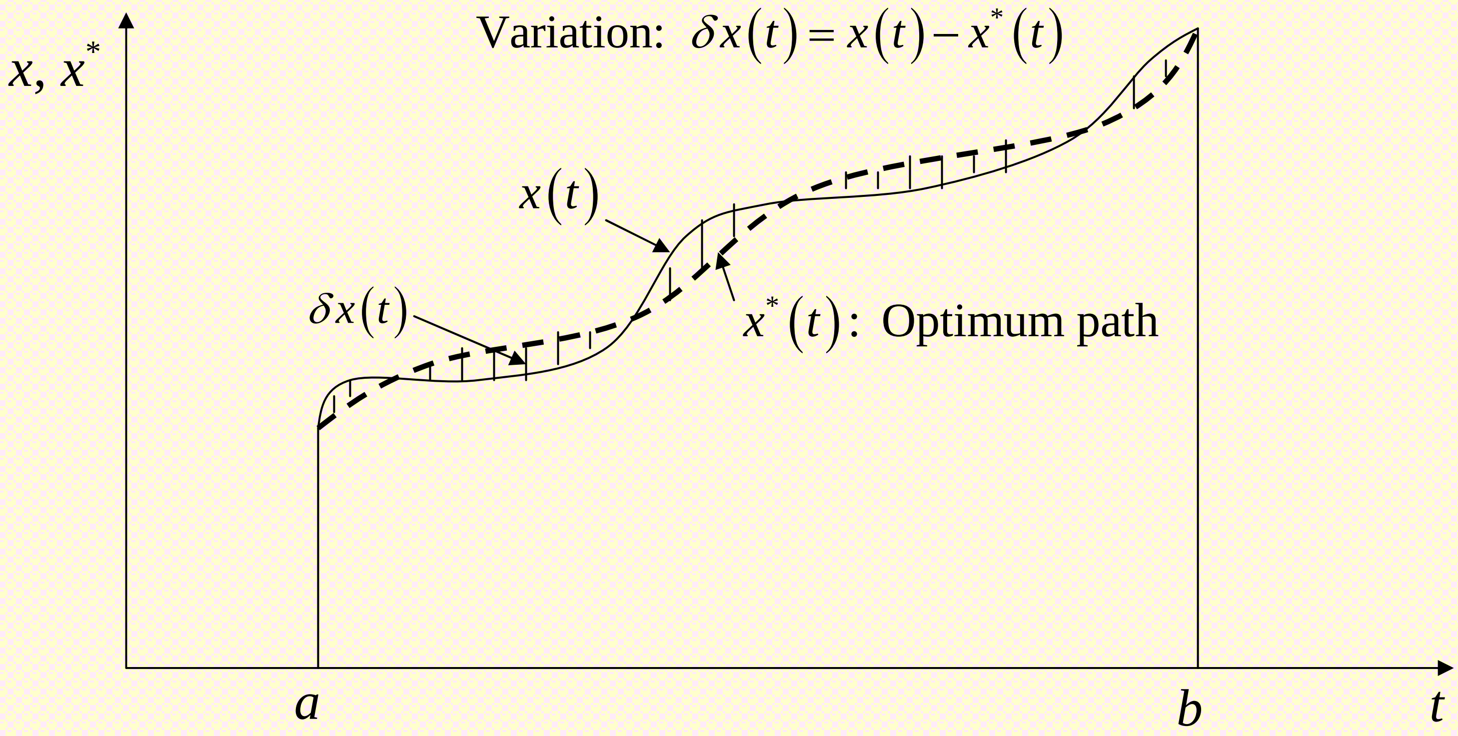
A functional is said to have a relative optimum at $x^*(t)$, if $\exists\, \varepsilon > 0$ such that for all functions $x(t) \in \Omega$ which satisfy $\left| x(t) - x^*(t) \right| < \varepsilon$, the increment of J has the "same sign".

1) If $\Delta J = J(x) - J(x^*) \geq 0$, then $J(x^*)$ is a relative (local) "Minimum".

2) If $\Delta J = J(x) - J(x^*) \leq 0$, then $J(x^*)$ is a relative (local) "Maximum".

Note: If the above relationships are satisfied for arbitrarily large $\varepsilon > 0$, then $J(x^*)$ is a "global optimum".

Optimum of a Functional



Fundamental Theorem of Calculus of Variations

For $x^*(t)$ to be a candidate for optimum, the following conditions hold good:

1) Necessary Condition: $\delta J\left(x^*(t), \delta x(t)\right) = 0, \quad \forall$ admissible $\delta x(t)$

2) Sufficiency Condition:

$$\delta^2 J > 0 \text{ (for minimum)}$$

$$\delta^2 J < 0 \text{ (for maximum)}$$

Fundamental Lemma

If for every continuous function $g(t)$

$$\int_{t_0}^{t_f} g(t)\,\delta x(t)\,dt = 0$$

where the variation $\delta x(t)$ is continuous in $t \in [t_0, t_f]$,

then

$$g(t) = 0 \quad \forall t \in [t_0, t_f]$$

Necessary condition of optimality

Problem: Optimize $J = \int_{t_0}^{t_f} L\left[x(t), \dot{x}(t), t\right] dt$ by appropriate selection of $x(t)$.

Note: t_0, t_f are fixed.

Solution: Make sure $\delta J = 0$ for arbitrary $\delta x(t)$

Necessary Conditions:

1) Euler – Lagrange (E-L) Equation

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0$$

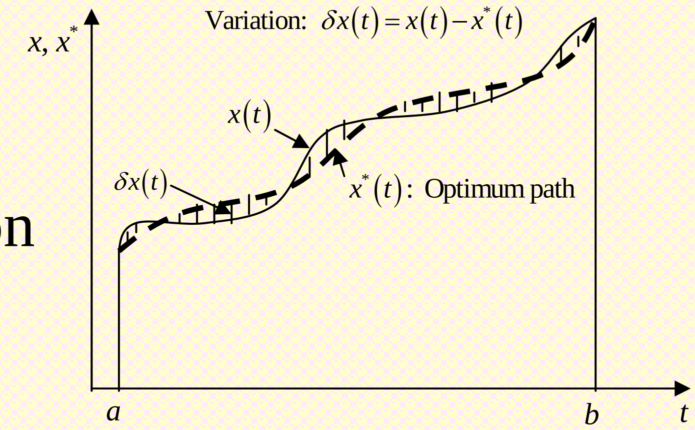
2) Transversality (Boundary) Condition

$$\left[\frac{\partial L}{\partial \dot{x}} \; \delta x\right]_{t_0}^{t_f} = 0$$

Note: Part of this equation may already be satisfied by problem formulation

Proof:

Hint: Use the necessary condition of optimality from the Fundamental theorem



Let $x^*(t), \quad t \in [t_0, t_f]$: Optimum solution

$x^*(t) + \delta x(t)$: Some adjacent solution

Then

$$\Delta J = J - J^* = \int_{t_0}^{t_f} L\left[x(t), \dot{x}(t), t\right] dt - \int_{t_0}^{t_f} L\left[x^*(t), \dot{x}^*(t), t\right] dt$$

$$= \int_{t_0}^{t_f} \underbrace{\left\{ L\left[x(t), \dot{x}(t), t\right] - L\left[x^*(t), \dot{x}^*(t), t\right] \right\}}_{\Delta L} dt = \int_{t_0}^{t_f} \Delta L \ dt$$

Proof

However at every point t,

$$\Delta L = L\left[x^* + \delta x,\ \dot{x}^* + \delta \dot{x},\ t\right] - L\left[x^*(t),\ \dot{x}^*(t),\ t\right]$$

$$= \left.\frac{\partial L}{\partial x}\right|_{x^*,\dot{x}_f^*} \delta x + \left.\frac{\partial L}{\partial \dot{x}}\right|_{x^*,\dot{x}_f^*} \delta \dot{x} + HOT$$

Assumption: L is continuous and smooth in both x and $\dot{x}$.

Then, in the limit, $\Delta L \;\to\; \delta L = \left[\dfrac{\partial L}{\partial x}\delta x + \dfrac{\partial L}{\partial \dot{x}}\delta \dot{x}\right]$. In that case,

$$\boxed{\Delta J \to \delta J = \int_{t_0}^{t_f} \left[\frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta \dot{x}\right] dt}$$

Proof

However,

$$\int_{t_0}^{t_f}\left[\frac{\partial L}{\partial \dot{x}}\delta\dot{x}\right]dt=\int_{t_0}^{t_f}\left[\left(\frac{\partial L}{\partial \dot{x}}\right)\frac{d(\delta x)}{dt}\right]dt$$

$$=\left[\left(\frac{\partial L}{\partial \dot{x}}\right)\int_{t_0}^{t_f}\frac{d(\delta x)}{dt}dt\right]_{t_o}^{t_f}-\int_{t_0}^{t_f}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\right]\int\left[\frac{d(\delta x)}{dt}dt\right]dt$$

$$=\left[\left(\frac{\partial L}{\partial \dot{x}}\right)\delta x\right]_{t_o}^{t_f}-\int_{t_0}^{t_f}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\right]\delta x\,dt$$

Proof

Hence,

$$\delta J = \int_{t_0}^{t_f} \left[\frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta \dot{x} \right] dt$$

$$= \int_{t_0}^{t_f} \left[\frac{\partial L}{\partial x} \right] \delta x\, dt + \left[\left(\frac{\partial L}{\partial \dot{x}} \right) \delta x \right]_{t_o}^{t_f} - \int_{t_0}^{t_f} \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} \right) \right] \delta x\, dt$$

$$= \int_{t_0}^{t_f} \left[\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} \right) \right] \delta x\, dt + \left[\left(\frac{\partial L}{\partial \dot{x}} \right) \delta x \right]_{t_o}^{t_f}$$

$$= 0 \quad \text{(Necessary condition of optimality)}$$

Necessary Conditions

1. $$\left(\frac{\partial L}{\partial x}\right)-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)=0 \qquad \text{(Euler-Lagrange Equation)}$$

2. $$\left[\left(\frac{\partial L}{\partial \dot{x}}\right)\delta x\right]_{t_o}^{t_f}=0 \qquad \text{(Transversality Condition)}$$

Note :

* Condition (1) must be satisfied regardless of the end condition.

* Part of second equation may already be satisfied by the problem of specification. i.e. the amount of extra information contained by this equation varies with the boundary conditions specified.

Example – 1

<u>Problem:</u> Minimize $J = \int_0^1 \left(\dot{x}^2 + x\right) dt$ with $x(0) = 2,\ x(1) = 3$

<u>Solution:</u> $L = \left(\dot{x}^2 + x\right)$

1) E-L Equation: $\dfrac{\partial L}{\partial x} - \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}}\right) = 0 \;\Rightarrow\; 1 - 2\ddot{x} = 0, \quad \ddot{x} = \dfrac{1}{2} \;\Rightarrow\; x(t) = \dfrac{t^2}{4} + c_1 t + c_2$

2) Boundary condition: $x(0) = c_2 = 2$

$$x(1) = \frac{1}{4} + c_1 + 2 = 3$$

$$c_1 = 1 - \frac{1}{4} = \frac{3}{4}$$

Hence, $x(t) = \dfrac{t^2}{4} + \dfrac{3t}{4} + 2$

Transversality condition is automatically satisfied, since $\delta x_0 = \delta x_f = 0$

Example – 2

<u>Problem:</u> Minimize $J = \int_0^1 \left(\dot{x}^2 + x \right) dt$ with $x(0) = 2,$ $\boxed{x(1):\ \text{Free}}$

<u>Solution:</u> $L = \left(\dot{x}^2 + x \right)$

1) E-L Equation: $\dfrac{\partial L}{\partial x} - \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}} \right) = 0 \;\Rightarrow\; 1 - 2\ddot{x} = 0, \quad \ddot{x} = \dfrac{1}{2} \;\Rightarrow\; x(t) = \dfrac{t^2}{4} + c_1 t + c_2$

2) Boundary condition: $x(0) = c_2 = 2$

$$\left. \frac{\partial L}{\partial \dot{x}} \right|_{t_f} \delta x_f = 0, \;\Rightarrow\; \left. \frac{\partial L}{\partial \dot{x}} \right|_{t_f} = 0 \quad \left(\because \delta x_f \neq 0 \right)$$

$$2\left. \dot{x} \right|_{t_f = 1} = \left. \frac{t_f}{2} \right|_{t_f = 1} + c_1 = 0, \;\Rightarrow\; c_1 = -\frac{1}{2}$$

Hence, $x(t) = \dfrac{t^2}{4} - \dfrac{t}{2} + 2$

Transversality Condition

General condition:

$$\boxed{\left[\frac{\partial L}{\partial \dot{x}}\;\delta x\right]_{t_0}^{t_f} + \left[\left\{L - \dot{x}\frac{\partial L}{\partial \dot{x}}\right\}\;\delta t\right]_{t_0}^{t_f} = 0}$$

Special Cases:

1) Fixed End Points: (t_0, x_0) and (t_f, x_f) are fixed.

No additional information!

2) t_0 and t_f are fixed (free initial and final states)

$$\left[\frac{\partial L}{\partial \dot{x}}\;\delta x\right]_{t_0}^{t_f} = 0$$

Transversality Condition

Special Cases:

3) t_0, x_0 are fixed (free final time, free final state)

$$\left.\frac{\partial L}{\partial \dot{x}}\right|_{t_f} \delta x_f + \left.\left\{L - \dot{x}\frac{\partial L}{\partial \dot{x}}\right\}\right|_{t_f} \delta t_f = 0$$

4) t_0, x_0 and x_f are fixed (free final time)

$$\left.\left\{L - \dot{x}\frac{\partial L}{\partial \dot{x}}\right\}\right|_{t_f} = 0$$

5) t_0, x_0 and t_f are fixed (free final state)

$$\left.\frac{\partial L}{\partial \dot{x}}\right|_{t_f} = 0$$

Transversality Condition

Special Cases:

6) (t_0, x_0) is fixed; (t_f, x_f) is constrained to lie on a given curve $\eta(t)$

$$\left.\frac{\partial L}{\partial \dot{x}}\right|_{t_f} \delta x_f + \left.\left\{L - \dot{x}\frac{\partial L}{\partial \dot{x}}\right\}\right|_{t_f} \delta t_f = 0. \quad \text{However,} \quad \delta x_f = \left.\frac{d\eta}{dt}\right|_{t_f} \delta t_f = \dot{\eta}_f \, \delta t_f$$

Hence, the transversality condition is

$$\left[\left.\frac{\partial L}{\partial \dot{x}}\right|_{t_f} \dot{\eta}_f + \left.\left\{L - \dot{x}\frac{\partial L}{\partial \dot{x}}\right\}\right|_{t_f}\right] \delta t_f = 0 \qquad (\delta t_f \neq 0)$$

Finally: $$\boxed{\left.\left\{L + (\dot{\eta} - \dot{x})\frac{\partial L}{\partial \dot{x}}\right\}\right|_{t_f} = 0}$$

Example

<u>Problem:</u> Minimize $J=\int_0^1\left(\sqrt{1+\dot{x}^2}\right)dt$ with $x(0)=0$ and $\left(t_f,x_f\right)$ lie on $y(t)=-5t+15$

<u>Solution:</u> $L=\sqrt{1+\dot{x}^2}$

1) E-L Equation: $$\frac{\partial L}{\partial x}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)=0$$

$$0-\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}\sqrt{1+\dot{x}^2}\right)=0$$

$$\frac{d}{dt}\left(\frac{\cancel{2}\dot{x}}{\cancel{2}\sqrt{1+\dot{x}^2}}\right)=0 \quad\Rightarrow\quad \frac{\sqrt{1+\dot{x}^2}\,\ddot{x}-\dot{x}\dfrac{2\,\dot{x}\,\ddot{x}}{2\sqrt{1+\dot{x}^2}}}{\left(1+\dot{x}^2\right)}=0$$

$$2\left(1+\cancel{\dot{x}^2}\right)\ddot{x}-\left(\cancel{2\,\dot{x}^2\,\ddot{x}}\right)=0$$

Example

1) E-L Equation: $\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \quad \Rightarrow \quad \ddot{x} = 0 \quad \Rightarrow \quad x(t) = c_1 t + c_2$

2) Boundary condition:

(1) $x(0) = c_2 = 0 \qquad \Rightarrow \qquad x(t) = c_1 t$

(2) $\left[L + (\dot{y} - \dot{x}) \frac{\partial L}{\partial \dot{x}} \right]\Big|_{t_f} = 0$

$$-5\dot{x}_f + 1 = 0 \qquad \Rightarrow \qquad -5c_1 + 1 = 0 \qquad \Rightarrow \qquad c_1 = 1/5$$

Hence, $x(t) = t/5$

To find t_f, $\quad t_f/5 = -5t_f + 15 \qquad \Rightarrow \qquad t_f = 75/26$

Variational Problems in Multiple Dimensions: Without Constraints

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Multiple Dimension Problems without constraints

Problem: Optimize $J = \int_{t_0}^{t_f} L\left[X(t), \dot{X}(t), t\right] dt$ by appropriate selection of $X(t)$.

where $X \triangleq \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$

Solution: Make sure $\delta J = 0$ for arbitrary $\delta X(t)$

Necessary Conditions:

1) Euler – Lagrange (E-L) Equation

$$\boxed{\frac{\partial L}{\partial X} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}}\right) = 0}$$

2) Transversality (Boundary) Condition

$$\boxed{\left[\left(\frac{\partial L}{\partial \dot{X}}\right)^T \delta X\right]_{t_0}^{t_f} + \left[\left\{L - \dot{X}^T\left(\frac{\partial L}{\partial \dot{X}}\right)\right\}\delta t\right]_{t_0}^{t_f} = 0}$$

Variational Problems with Constraints

$$\text{Optimize}: J = \int_{t_0}^{t_f} L\left(X, \dot{X}, t\right) dt$$

$$\text{Subject to}: \quad \Phi\left(X, \dot{X}, t\right) = 0$$

where

$$X \triangleq \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T, \quad \Phi \triangleq \begin{bmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_{\tilde{n}} \end{bmatrix}^T$$

Variational Problems with Constraints

Lagrange's Existence Theorem:

$\exists\ \lambda_{\tilde{n}\times 1}(t)$: The above constrained optimization problem leads to the same solution as the following unconstrained cost functional

$$\bar{J} = \int_{t_0}^{t_f} \left[L\left(X, \dot{X}, t \right) + \lambda^T \Phi\left(X, \dot{X}, t \right) \right] dt$$

Let $L^*\left(X, \dot{X}, t \right) = L\left(X, \dot{X}, t \right) + \lambda^T \Phi\left(X, \dot{X}, t \right)$

Variational Problems with Constraints

Necessary Conditions of Optimality:

(1) E-L Equations:

$$\text{(a)}\ \frac{\partial L^*}{\partial X} - \frac{d}{dt}\left[\frac{\partial L^*}{\partial \dot{X}}\right] = 0 \ \ (\text{ n equations})$$

$$\text{(b)}\ \frac{\partial L^*}{\partial \lambda} - \frac{d}{dt}\cancel{\left[\frac{\partial L^*}{\partial \dot{\lambda}}\right]} = 0 \ \ (\ \tilde{n} \text{ equations})$$

$$\left(\text{Note: } \frac{\partial L^*}{\partial \dot{\lambda}} = 0 \text{ as there is no } \dot{\lambda} \text{ term in } L^*\right)$$

Variational Problems with Constraints

(2) Transversality Conditions:

$$\text{(a)}\quad \left[\left(\frac{\partial L^*}{\partial \dot{X}}\right)^T \delta X\right]_{t_o}^{t_f} + \left[\left\{L^* - \dot{X}^T\left(\frac{\partial L^*}{\partial \dot{X}}\right)\right\}\delta t\right]_{t_o}^{t_f} = 0$$

$$\text{(b)}\quad \left[\cancel{\left(\frac{\partial L^*}{\partial \dot{\lambda}}\right)}^T \delta \lambda\right]_{t_o}^{t_f} + \left[\left\{L^* - \dot{\lambda}^T\cancel{\left(\frac{\partial L^*}{\partial \dot{\lambda}}\right)}\right\}\delta t\right]_{t_o}^{t_f} = 0;$$

Variational Problems with Constraints

E-L Equations:

Varaibles: $n \quad + \quad \tilde{n} \quad + \; 1$
$(X) \quad (\lambda) \quad (t_f)$
Boundary Conditions : $n \; + \; \tilde{n} \; + \; 1$

1) (a) $\left(\dfrac{\partial L^*}{\partial X}\right) - \dfrac{d}{dt}\left(\dfrac{\partial L^*}{\partial \dot{X}}\right) = 0$

(b) $\left(\dfrac{\partial L^*}{\partial \lambda}\right) = \Phi\left(X, \dot{X}, t\right) = 0$ (same constraint equation)

2) Transversality Conditions: $\left(t_0, X_0\right)$ fixed, $\left(t_f, X_f\right)$ free

(a) $\left(\dfrac{\partial L^*}{\partial \dot{X}}\right)_{t_f}^T \delta X_f + \left[L^* - \dot{X}^T \left(\dfrac{\partial L^*}{\partial \dot{X}}\right)\right]_{t_f} \delta t_f = 0 \qquad$ ($\tilde{n}$ equations)

(b) $L^*_{t_f} \delta t_f = 0 \qquad$ However t_f is free $\Rightarrow \quad \delta t_f \neq 0$

so $\quad L^*_{t_f} = 0 \qquad$ (1 equation)

Constraint Equations

- Nonholonomic constraints

$$\Phi\left(X, \dot{X}, t \right) = 0$$

- Isoperimetric constraints

$$\int_{t_0}^{t_f} q\left(X, \dot{X}, t \right) dt = k$$

One way to get rid of Isoperimetric constraints is to convert them into Nonholonomic constraints.

Isoperimetric Constraints

Define: $\dot{x}_{n+1} = q\left(X, \dot{X}, t\right)$

Then

$$\int_{t_0}^{t_f} \dot{x}_{n+1} dt = k$$

$$x_{n+1}(t_f) - x_{n+1}(t_0) = k$$

Choose one of $x_{n+1}(t_f)$ or $x_{n+1}(t_0)$ and fix the other

Let $x_{n+1}(t_0) = 0$

$$x_{n+1}(t_f) = k$$

Isoperimetric Constraints

Summary :

The following additional non-holonomic constraint is introduced:

$$\dot{x}_{n+1} = q\left(X, \dot{X}, t \right)$$

with boundary conditions:

$$x_{n+1}(t_0) = 0$$
$$x_{n+1}(t_f) = k$$

The original problem is augmented with this information and solved.

References

- T. F. Elbert, *Estimation and Control Systems*, Von Nostard Reinhold, 1984.
- D. S. Naidu, *Optimal Control Systems*, CRC Press, 2002.

Thanks for the Attention…!