# Module
# 1
# Energy Methods in Structural Analysis

# Lesson
# 3
# Castigliano's Theorems

## Instructional Objectives

After reading this lesson, the reader will be able to;

1. State and prove first theorem of Castigliano.
2. Calculate deflections along the direction of applied load of a statically determinate structure at the point of application of load.
3. Calculate deflections of a statically determinate structure in any direction at a point where the load is not acting by fictious (imaginary) load method.
4. State and prove Castigliano's second theorem.

## 3.1 Introduction

In the previous chapter concepts of strain energy and complementary strain energy were discussed. Castigliano's first theorem is being used in structural analysis for finding deflection of an elastic structure based on strain energy of the structure. The Castigliano's theorem can be applied when the supports of the structure are unyielding and the temperature of the structure is constant.

## 3.2 Castigliano's First Theorem

For linearly elastic structure, where external forces only cause deformations, the complementary energy is equal to the strain energy. For such structures, the Castigliano's first theorem may be stated as the first partial derivative of the strain energy of the structure with respect to any particular force gives the displacement of the point of application of that force in the direction of its line of action.

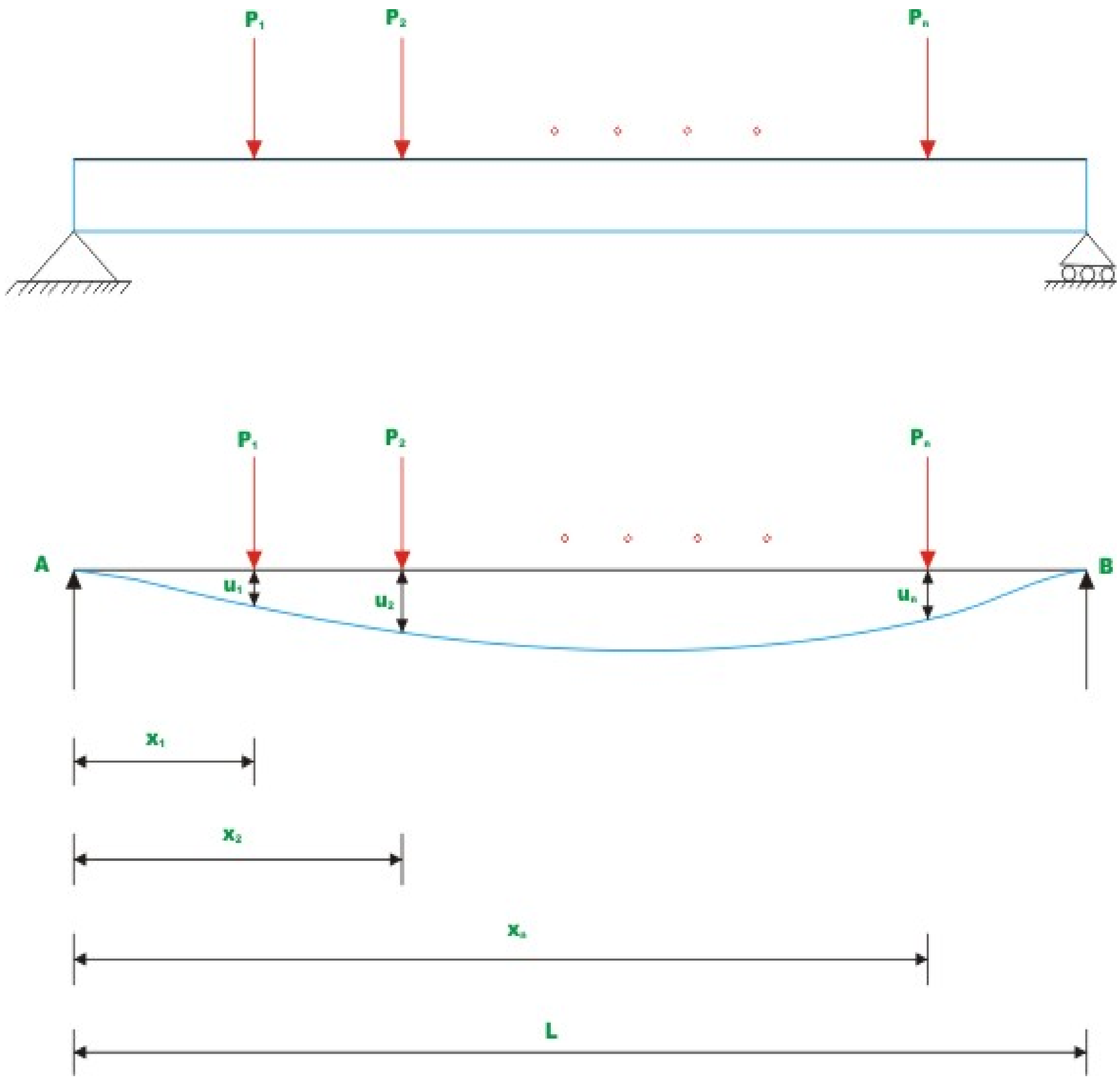


**Fig. 3.1 Castigliano's First Theorem**

Let $P_1, P_2, ...., P_n$ be the forces acting at $x_1, x_2, ......, x_n$ from the left end on a simply supported beam of span $L$. Let $u_1, u_2, ..., u_n$ be the displacements at the loading points $P_1, P_2, ...., P_n$ respectively as shown in Fig. 3.1. Now, assume that the material obeys Hooke's law and invoking the principle of superposition, the work done by the external forces is given by (vide eqn. 1.8 of lesson 1)

$$W = \frac{1}{2}P_1 u_1 + \frac{1}{2}P_2 u_2 + ......... + \frac{1}{2}P_n u_n \qquad (3.1)$$

Work done by the external forces is stored in the structure as strain energy in a conservative system. Hence, the strain energy of the structure is,

$$U = \frac{1}{2}P_1u_1 + \frac{1}{2}P_2u_2 + .........+ \frac{1}{2}P_nu_n \tag{3.2}$$

Displacement $u_1$ below point $P_1$ is due to the action of $P_1, P_2, ...., P_n$ acting at distances $x_1, x_2, ......., x_n$ respectively from left support. Hence, $u_1$ may be expressed as,

$$u_1 = a_{11}P_1 + a_{12}P_2 + .........+ a_{1n}P_n \tag{3.3}$$

In general,

$$u_i = a_{i1}P_1 + a_{i2}P_2 + .........+ a_{in}P_n \qquad i = 1,2,...n \tag{3.4}$$

where $a_{ij}$ is the flexibility coefficient at $i$ due to unit force applied at $j$. Substituting the values of $u_1, u_2, ..., u_n$ in equation (3.2) from equation (3.4), we get,

$$U = \frac{1}{2}P_1[a_{11}P_1 + a_{12}P_2 + ...] + \frac{1}{2}P_2[a_{21}P_1 + a_{22}P_2 + ...] + .......+ \frac{1}{2}P_n[a_{n1}P_1 + a_{n2}P_2 + ...] \tag{3.5}$$

We know from Maxwell-Betti's reciprocal theorem $a_{ij} = a_{ji}$. Hence, equation (3.5) may be simplified as,

$$U = \frac{1}{2}\left[a_{11}P_1^2 + a_{22}P_2^2 + ....+ a_{nn}P_n^2\right] + \left[a_{12}P_1P_2 + a_{13}P_1P_3 + ....+ a_{1n}P_1P_n\right] + ... \tag{3.6}$$

Now, differentiating the strain energy with any force $P_1$ gives,

$$\frac{\partial U}{\partial P_1} = a_{11}P_1 + a_{12}P_2 + .........+ a_{1n}P_n \tag{3.7}$$

It may be observed that equation (3.7) is nothing but displacement $u_1$ at the loading point.
In general,

$$\frac{\partial U}{\partial P_n} = u_n \tag{3.8}$$

Hence, for determinate structure within linear elastic range the partial derivative of the total strain energy with respect to any external load is equal to the displacement of the point of application of load in the direction of the applied load, provided the supports are unyielding and temperature is maintained constant. This theorem is advantageously used for calculating deflections in elastic structure. The procedure for calculating the deflection is illustrated with few examples.

**Example 3.1**

Find the displacement and slope at the tip of a cantilever beam loaded as in Fig. 3.2. Assume the flexural rigidity of the beam $EI$ to be constant for the beam.

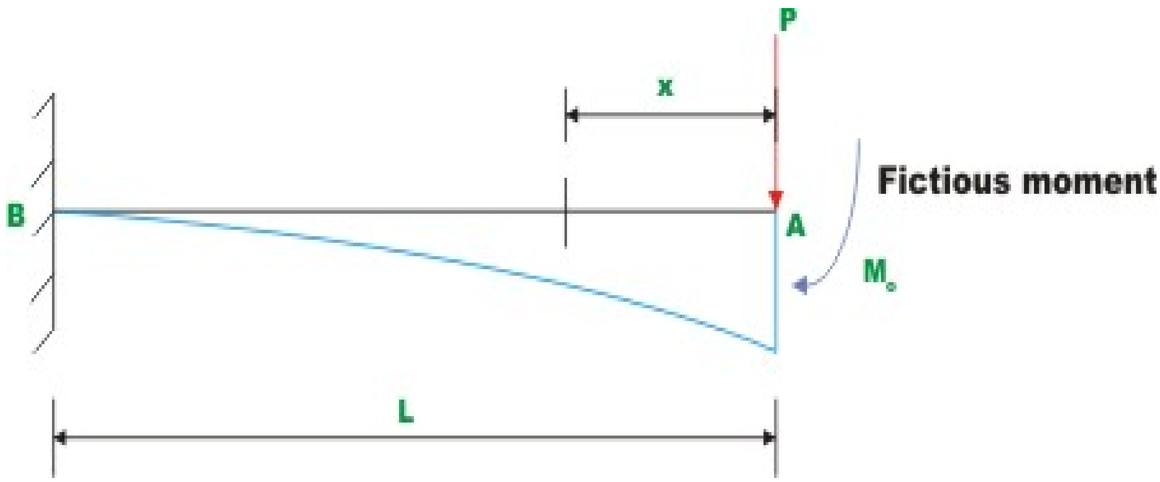


**Fig. 3.2 Example 3.1**

Moment at any section at a distance $x$ away from the free end is given by

$$M = -Px \tag{1}$$

Strain energy stored in the beam due to bending is $U = \int_0^L \frac{M^2}{2EI} dx$ (2)

Substituting the expression for bending moment $M$ in equation (3.10), we get,

$$U = \int_0^L \frac{(Px)^2}{2EI} dx = \frac{P^2 L^3}{6EI} \tag{3}$$

Now, according to Castigliano's theorem, the first partial derivative of strain energy with respect to external force $P$ gives the deflection $u_A$ at A in the direction of applied force. Thus,

$$\frac{\partial U}{\partial P} = u_A = \frac{PL^3}{3EI} \tag{4}$$

To find the slope at the free end, we need to differentiate strain energy with respect to externally applied moment $M$ at $A$. As there is no moment at $A$, apply a fictitious moment $M_0$ at $A$. Now moment at any section at a distance $x$ away from the free end is given by

$$M = -Px - M_0$$

Now, strain energy stored in the beam may be calculated as,

$$U = \int_0^L \frac{(Px + M_0)^2}{2EI} dx = \frac{P^2L^3}{6EI} + \frac{M_0PL^2}{2EI} + \frac{{M_0}^2 L}{2EI} \tag{5}$$

Taking partial derivative of strain energy with respect to $M_0$, we get slope at $A$.

$$\frac{\partial U}{\partial M_0} = \theta_A = \frac{PL^2}{2EI} + \frac{M_0 L}{EI} \tag{6}$$

But actually there is no moment applied at $A$. Hence substitute $M_0 = 0$ in equation (3.14) we get the slope at A.

$$\theta_A = \frac{PL^2}{2EI} \tag{7}$$

**Example 3.2**

A cantilever beam which is curved in the shape of a quadrant of a circle is loaded as shown in Fig. 3.3. The radius of curvature of curved beam is $R$, Young's modulus of the material is $E$ and second moment of the area is $I$ about an axis perpendicular to the plane of the paper through the centroid of the cross section. Find the vertical displacement of point $A$ on the curved beam.

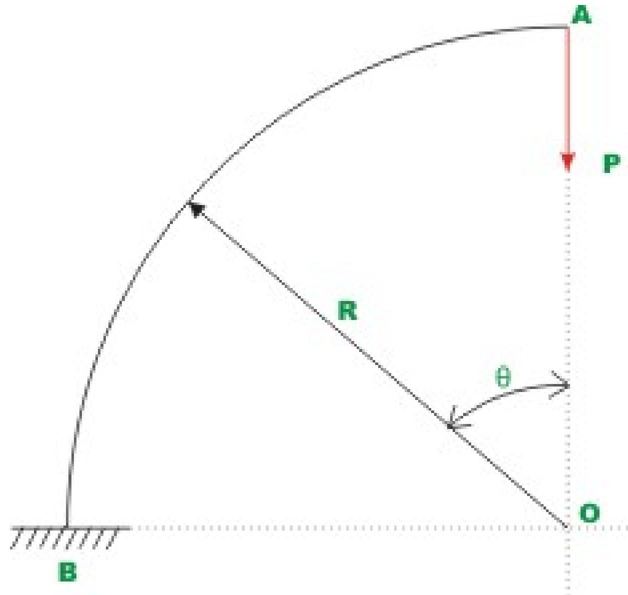


**Fig. 3.3 Example 3.2**

The bending moment at any section $\theta$ of the curved beam (see Fig. 3.3) is given by

$$M = PR\sin\theta \quad (1)$$

Strain energy $U$ stored in the curved beam due to bending is,

$$U = \int_0^s \frac{M^2}{2EI} ds = \int_0^{\pi/2} \frac{P^2R^2(\sin^2\theta)Rd\theta}{2EI} = \frac{P^2R^3}{2EI}\frac{\pi}{4} = \frac{\pi P^2R^3}{8EI} \quad (2)$$

Differentiating strain energy with respect to externally applied load, $P$ we get

$$u_A = \frac{\partial U_b}{\partial P} = \frac{\pi PR^3}{4EI} \quad (3)$$

## Example 3.3

Find horizontal displacement at D of the frame shown in Fig. 3.4. Assume the flexural rigidity of the beam $EI$ to be constant through out the member. Neglect strain energy due to axial deformations.

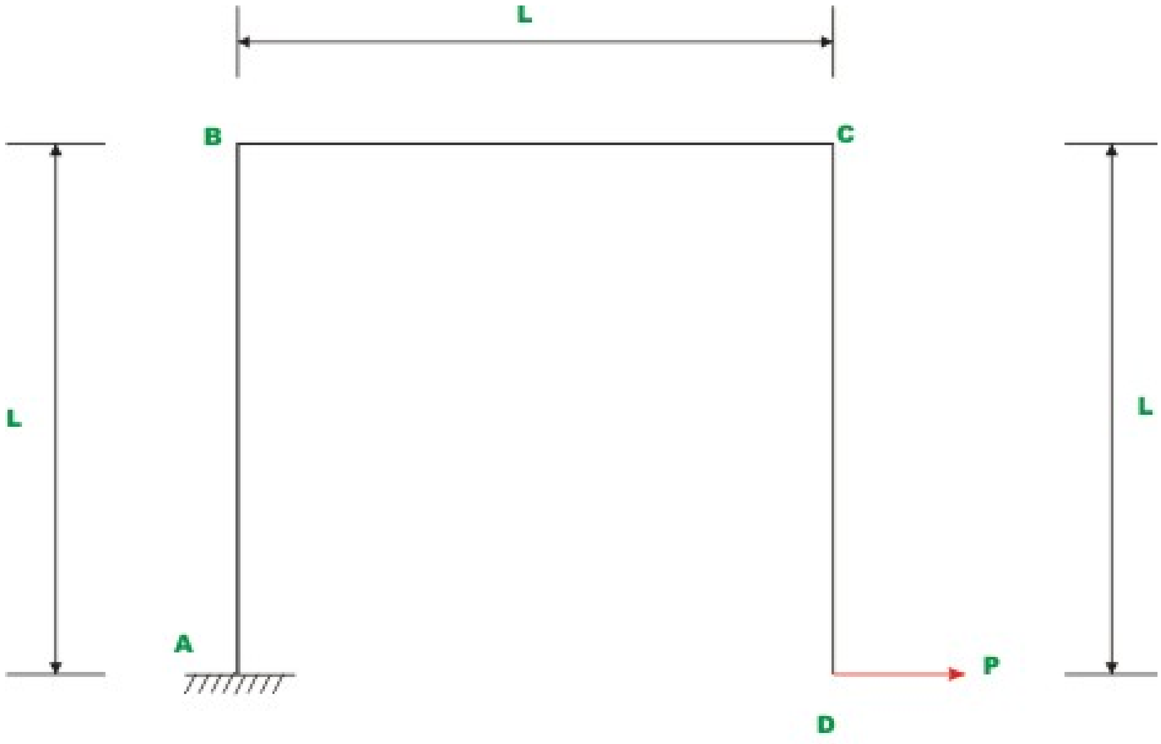


**Fig. 3.4 Example 3.3**

The deflection D may be obtained via. Castigliano's theorem. The beam segments $BA$ and $DC$ are subjected to bending moment $Px$ $(0 < x < L)$ and the beam element BC is subjected to a constant bending moment of magnitude $PL$.

Total strain energy stored in the frame due to bending

$$U = 2\int_0^L \frac{(Px)^2}{2EI} dx + \int_0^L \frac{(PL)^2}{2EI} dx \quad (1)$$

After simplifications,

$$U = \frac{P^2 L^3}{3EI} + \frac{P^2 L^3}{2EI} = \frac{5P^2 L^3}{6EI} \quad (2)$$

Differentiating strain energy with respect to $P$ we get,

$$\frac{\partial U}{\partial P} = u_D = 2\frac{5P\ L^3}{6EI} = \frac{5P\ L^3}{3EI}$$

## Example 3.4

Find the vertical deflection at $A$ of the structure shown Fig. 3.5. Assume the flexural rigidity $EI$ and torsional rigidity $GJ$ to be constant for the structure.

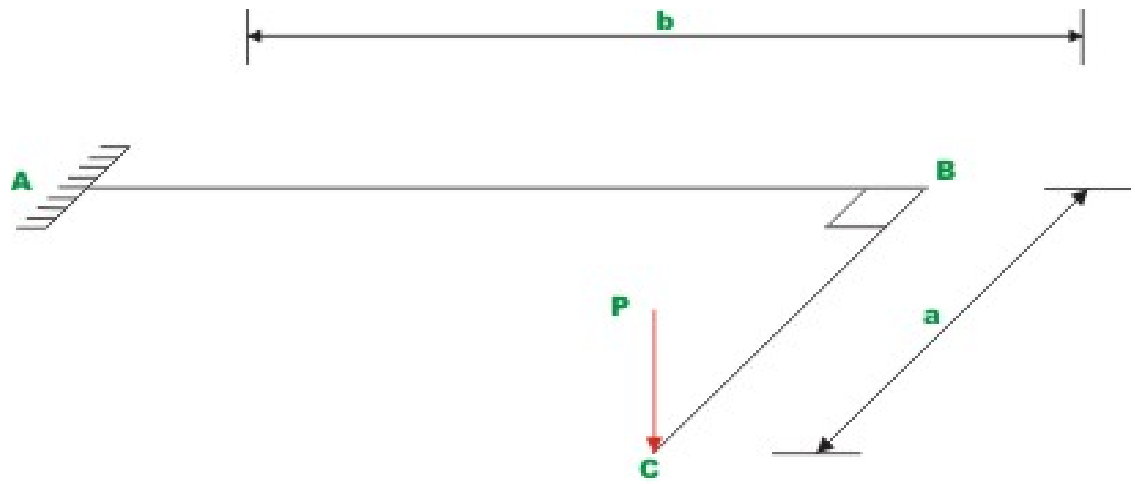


**Fig.3.5 Example 3.4**

The beam segment $BC$ is subjected to bending moment $Px$ ($0 < x < a$; x is measured from $C$ )and the beam element $AB$ is subjected to torsional moment of magnitude $Pa$ and a bending moment of $Px\,(0 \le x \le b\,;\text{x is measured from B})$. The strain energy stored in the beam $ABC$ is,

$$U = \int_0^a \frac{M^2}{2EI}dx + \int_0^b \frac{(Pa)^2}{2GJ}dx + \int_0^b \frac{(Px)^2}{2EI}dx \tag{1}$$

After simplifications,

$$U = \frac{P^2a^3}{6EI} + \frac{P^2a^2b}{2GJ} + \frac{P^2b^3}{6EI} \tag{2}$$

Vertical deflection $u_A$ at $A$ is,

$$\frac{\partial U}{\partial P} = u_A = \frac{Pa^3}{3EI} + \frac{Pa^2b}{GJ} + \frac{Pb^3}{3EI} \tag{3}$$

## Example 3.5

Find vertical deflection at $C$ of the beam shown in Fig. 3.6. Assume the flexural rigidity $EI$ to be constant for the structure.


**Fig. 3.6 Example 3.5**

The beam segment $CB$ is subjected to bending moment $Px$ $(0 < x < a)$ and beam element $AB$ is subjected to moment of magnitude $Pa$.

To find the vertical deflection at $C$, introduce a imaginary vertical force $Q$ at $C$. Now, the strain energy stored in the structure is,

$$U = \int_0^a \frac{(Px)^2}{2EI} dx + \int_0^b \frac{(Pa + Qy)^2}{2EI} dy \quad (1)$$

Differentiating strain energy with respect to $Q$, vertical deflection at $C$ is obtained.

$$\frac{\partial U}{\partial Q} = u_C = \int_0^b \frac{2(Pa + Qy)y}{2EI} dy \quad (2)$$

$$u_C = \frac{1}{EI} \int_0^b Pay + Qy^2 dy \quad (3)$$

$$u_C = \frac{1}{EI}\left[\frac{Pab^2}{2} + \frac{Qb^3}{3}\right] \quad (4)$$

But the force $Q$ is fictitious force and hence equal to zero. Hence, vertical deflection is,

$$u_C = \frac{Pab^2}{2EI} \quad (5)$$

## 3.3 Castigliano's Second Theorem

In any elastic structure having $n$ independent displacements $u_1, u_2, ..., u_n$ corresponding to external forces $P_1, P_2, ...., P_n$ along their lines of action, if strain energy is expressed in terms of displacements then $n$ equilibrium equations may be written as follows.

$$\frac{\partial U}{\partial u_j} = P_j, \quad j = 1, 2, ..., n \quad (3.9)$$

This may be proved as follows. The strain energy of an elastic body may be written as

$$U = \frac{1}{2}P_1 u_1 + \frac{1}{2}P_2 u_2 + .......... + \frac{1}{2}P_n u_n \quad (3.10)$$

We know from Lesson 1 (equation 1.5) that

$$P_i = k_{i1}u_1 + k_{i2}u_2 + ..... + k_{in}u_n, \qquad i = 1, 2, .., n \quad (3.11)$$

where $k_{ij}$ is the stiffness coefficient and is defined as the force at $i$ due to unit displacement applied at $j$. Hence, strain energy may be written as,

$$U = \frac{1}{2}u_1[k_{11}u_1 + k_{12}u_2 + ...] + \frac{1}{2}u_2[k_{21}u_1 + k_{22}u_2 + ...] + ....... + \frac{1}{2}u_n[k_{n1}u_1 + k_{n2}u_2 + ...] \quad (3.12)$$

We know from reciprocal theorem $k_{ij} = k_{ji}$. Hence, equation (3.12) may be simplified as,

$$U = \frac{1}{2}\left[k_{11}u_1^2 + k_{22}u_2^2 + .... + k_{nn}u_n^2\right] + \left[k_{12}u_1u_2 + k_{13}u_1u_3 + .... + k_{1n}u_1u_n\right] + ... \quad (3.13)$$

Now, differentiating the strain energy with respect to any displacement $u_1$ gives the applied force $P_1$ at that point, Hence,

$$\frac{\partial U}{\partial u_1} = k_{11}u_1 + k_{12}u_2 + ........ + k_{1n}u_n \tag{3.14}$$

Or,

$$\frac{\partial U}{\partial u_j} = P_j, \qquad j = 1, 2, ..., n \tag{3.15}$$

## Summary

In this lesson, Castigliano's first theorem has been stated and proved for linearly elastic structure with unyielding supports. The procedure to calculate deflections of a statically determinate structure at the point of application of load is illustrated with examples. Also, the procedure to calculate deflections in a statically determinate structure at a point where load is applied is illustrated with examples. The Castigliano's second theorem is stated for elastic structure and proved in section 3.4.