3.2 Analysis of Members under Flexure (Part I)

This section covers the following topics.

- Introduction
- Analyses at Transfer and at Service

3.2.1 Introduction

Similar to members under axial load, the analysis of members under flexure refers to the evaluation of the following.

1) Permissible prestress based on allowable stresses at transfer.
2) Stresses under service loads. These are compared with allowable stresses under service conditions.
3) Ultimate strength. This is compared with the demand under factored loads.
4) The entire load versus deformation behaviour.

The analyses at transfer and under service loads are presented in this section. The analysis for the ultimate strength is presented separately in Section 3.4, Analysis of Member under Flexure (Part III). The evaluation of the load versus deformation behaviour is required in special type of analysis. This analysis will not be covered in this section.

Assumptions

The analysis of members under flexure considers the following.

1) Plane sections remain plane till failure (known as Bernoulli's hypothesis).
2) Perfect bond between concrete and prestressing steel for bonded tendons.

Principles of Mechanics

The analysis involves three principles of mechanics.

1) **Equilibrium** of internal forces with the external loads. The compression in concrete (*C*) is equal to the tension in the tendon (*T*). The couple of C and T are equal to the moment due to external loads.
2) **Compatibility** of the strains in concrete and in steel for bonded tendons. The formulation also involves the first assumption of plane section remaining plane after bending. For unbonded tendons, the compatibility is in terms of deformation.

3) **Constitutive** relationships relating the stresses and the strains in the materials.

Variation of Internal Forces

In reinforced concrete members under flexure, the values of compression in concrete (C) and tension in the steel (T) increase with increasing external load. The change in the lever arm (z) is not large.

In prestressed concrete members under flexure, at transfer of prestress C is located close to T. The couple of C and T balance only the self weight. At service loads, C shifts up and the lever arm (z) gets large. The variation of C or T is not appreciable.

The following figure explains this difference schematically for a simply supported beam under uniform load.

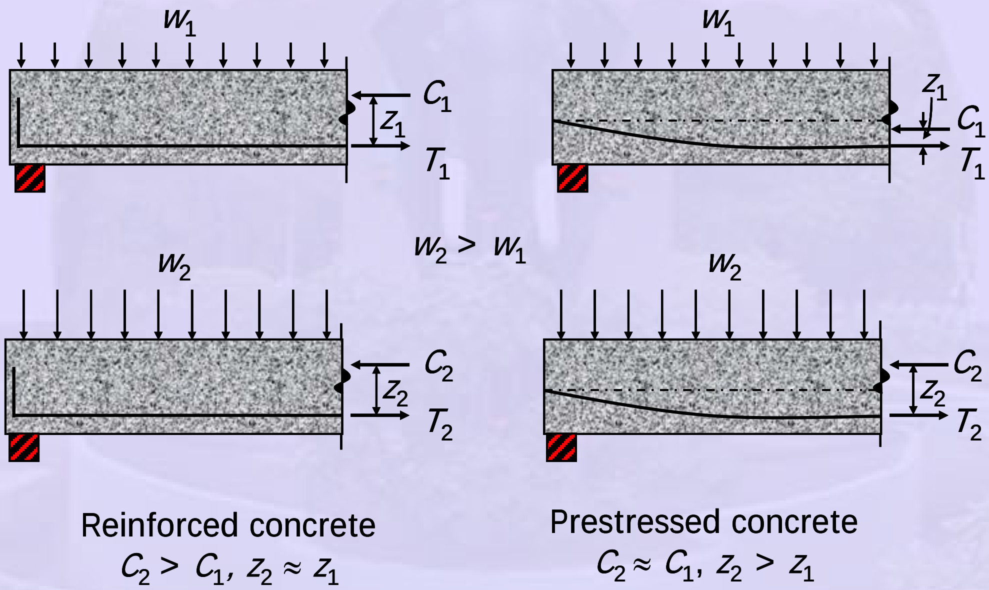


Figure 3-2.1 Variations of internal forces and lever arms

In the above figure,

C_1, T_1 = compression and tension at transfer due to self weight

C_2, T_2 = compression and tension under service loads

w_1 = self weight

w_2 = service loads

z_1 = lever arm at transfer

z_2 = lever arm under service loads.

For the reinforced concrete member C_2 is substantially large than C_1, but z_2 is close to z_1. For the prestressed concrete member C_2 is close to C_1, but z_2 is substantially large than z_1.

3.2.2 Analyses at Transfer and at Service

The analyses at transfer and under service loads are similar. Hence, they are presented together. A prestressed member usually remains uncracked under service loads. The concrete and steel are treated as elastic materials. The principle of superposition is applied. The increase in stress in the prestressing steel due to bending is neglected.

There are three approaches to analyse a prestressed member at transfer and under service loads. These approaches are based on the following concepts.

a) Based on stress concept.
b) Based on force concept.
c) Based on load balancing concept.

The following material explains the three concepts.

Based on Stress Concept

In the approach based on stress concept, the stresses at the edges of the section under the internal forces in concrete are calculated. The stress concept is used to compare the calculated stresses with the allowable stresses.

The following figure shows a simply supported beam under a uniformly distributed load (UDL) and prestressed with constant eccentricity (e) along its length.

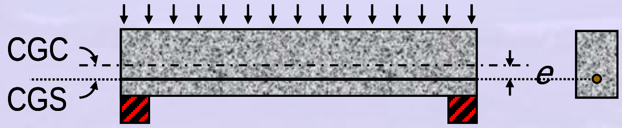


Figure 3-2.2 A simply supported beam under UDL

The following sketch shows the internal forces in concrete at a section and the corresponding stress profiles. The first stress profile is due to the compression P. The second profile is due to the eccentricity of the compression. The third profile is due to the moment. At transfer, the moment is due to self weight. At service the moment is due to service loads.

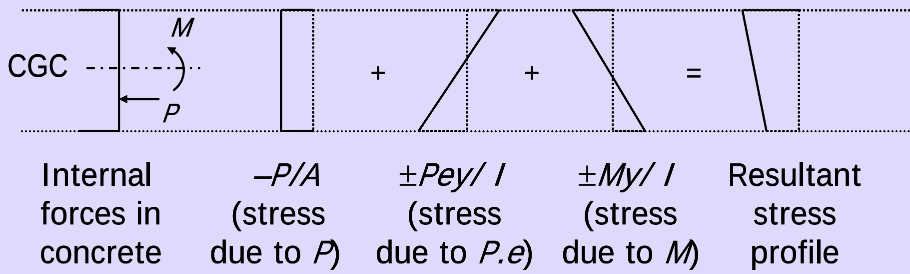


Figure 3-2.3 Stress profiles at a section due to internal forces

The resultant stress at a distance *y* from the CGC is given by the principle of superposition as follows.

$$f = -\frac{P}{A} \pm \frac{Pey}{I} \pm \frac{My}{I} \qquad \textbf{(3-2.1)}$$

For a curved tendon, *P* can be substituted by its horizontal component. But the effect of the refinement is negligible.

Based on Force Concept

The approach based on force concept is analogous to the study of reinforced concrete. The tension in prestressing steel (*T*) and the resultant compression in concrete (*C*) are considered to balance the external loads. This approach is used to determine the dimensions of a section and to check the service load capacity. Of course, the stresses in concrete calculated by this approach are same as those calculated based on stress concept. The stresses at the extreme edges are compared with the allowable stresses.

The following figures show the internal forces in the section.

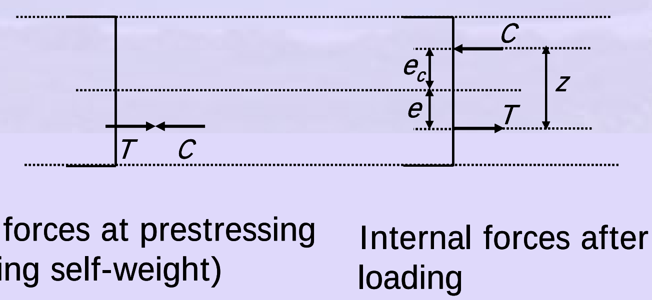


Figure 3-2.4 Internal forces at a section

The equilibrium equations are as follows.

$$C = T$$

$$M = C.z \tag{3-2.2}$$

$$M = C(e_c + e) \tag{3-2.3}$$

The resultant stress in concrete at distance *y* from the CGC is given as follows.

$$f = -\frac{C}{A} \pm \frac{Ce_c y}{I} \tag{3-2.4}$$

Substituting $C = P$ and $Ce_c = M – Pe$, the expression of stress becomes same as that given by the stress concept.

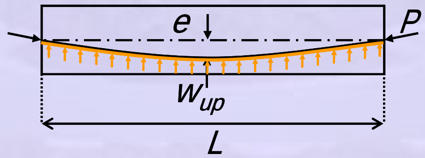
$$f = -\frac{P}{A} \pm \frac{Pey}{I} \pm \frac{My}{I} \tag{3-2.5}$$

Based on Load Balancing Concept

The approach based on load balancing concept is used for a member with curved or harped tendons and in the analysis of indeterminate continuous beams. The moment, upward thrust and upward deflection (camber) due to the prestress in the tendons are calculated. The upward thrust balances part of the superimposed load.

The expressions for three profiles of tendons in simply supported beams are given.

a) For a Parabolic Tendon



Free body diagram of concrete

Bending moment diagram

Figure 3-2.5 Simply supported beam with parabolic tendon

The moment at the centre due to the uniform upward thrust (w_{up}) is given by the following equation.

$$M = \frac{w_{up}L^2}{8}$$

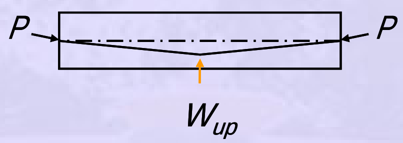
(3-2.6)

The moment at the centre from the prestressing force is given as $M = Pe$. The expression of w_{up} is calculated by equating the two expressions of M. The upward deflection (Δ) can be calculated from w_{up} based on elastic analysis.

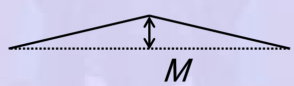
$$w_{up} = \frac{8Pe}{L^2}$$

$$\Delta = \frac{5w_{up}L^4}{384EI} \quad \textbf{(3-2.7)}$$

b) For Singly Harped Tendon



Free body diagram of concrete



Bending moment diagram

Figure 3-2.6 Simply supported beam with singly harped tendon

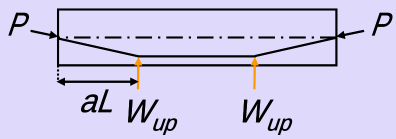
The moment at the centre due to the upward thrust (W_{up}) is given by the following equation. It is equated to the moment due to the eccentricity of the tendon. As before, the upward thrust and the deflection can be calculated.

$$M = \frac{W_{up}L}{4} = Pe$$

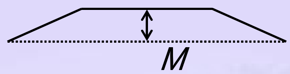
$$W_{up} = \frac{4Pe}{L}$$

$$\Delta = \frac{W_{up}L^3}{48EI} \quad \textbf{(3-2.8)}$$

c) For Doubly Harped Tendon



Free body diagram of concrete



Bending moment diagram

Figure 3-2.7 Simply supported beam with doubly harped tendon

The moment at the centre due to the upward thrusts (W_{up}) is given by the following equation. It is equated to the moment due to the eccentricity of the tendon. As before, the upward thrust and the deflection can be calculated.

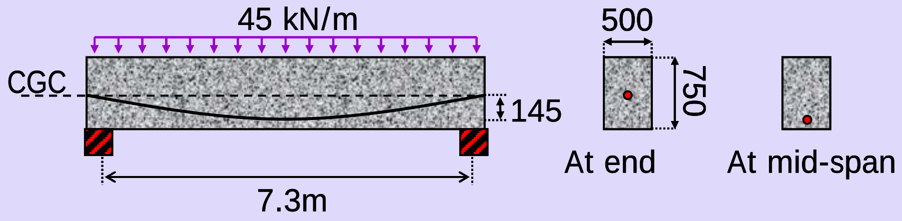
$$M = W_{up} aL = Pe$$

$$W_{up} = \frac{Pe}{aL}$$

$$\Delta = \frac{a\left(3-4a^2\right)W_{up}L^3}{24EI} \qquad \textbf{(3-2.9)}$$

Example 3-2.1

A concrete beam prestressed with a parabolic tendon is shown in the figure. The prestressing force applied is 1620 kN. The uniformly distributed load includes the self weight. Compute the extreme fibre stress at the mid-span by applying the three concepts. Draw the stress distribution across the section at mid-span.



Solution

a) Stress concept

Area of concrete, A = 500 × 750

= 375,000 mm^2

Moment of inertia, I = (500 × 750^3) / 12

= 1.758 × 10^{10} mm^4

Bending moment at mid-span, M = (45 × 7.32) / 8

= 299.7 kNm

Top fibre stress

$$(f_c)_t = -\frac{P}{A} + \frac{Pe}{I}y_{top} - \frac{M}{I}y_{top}$$

$$= -\frac{1620\times10^3}{375\times10^3} + \frac{1620\times10^3\times145}{1.758\times10^{10}}\times375 - \frac{299.7\times10^6}{1.758\times10^{10}}\times375$$

$$= -4.32 + 5.01 - 6.39$$

$$= -5.7\text{ N/mm}^2$$

Bottom fibre stress

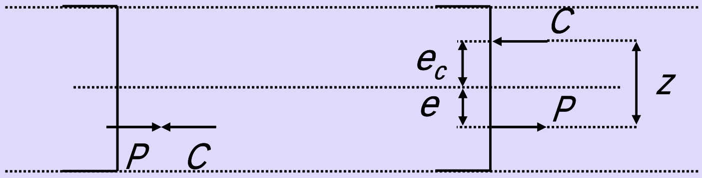
$$(f_c)_b = -\frac{P}{A} - \frac{Pe}{I}y_{bot} + \frac{M}{I}y_{bot}$$

$$= -\frac{1620\times10^3}{375\times10^3} - \frac{1620\times10^3\times145}{1.758\times10^{10}}\times375 + \frac{299.7\times10^6}{1.758\times10^{10}}\times375$$

$$= -4.32 - 5.01 + 6.39$$

$$= -2.9\text{ N/mm}^2$$

b) Force concept



Applied moment M = 299.7 kN-m

Lever arm z $= M / P$

$= 299.7 \times 10^3 / 1620$

$= 185$ mm

Eccentricity of C e_c $= z - e$

$= 185 - 145$

$= 40$ mm

Top fibre stress

$$(f_c)_t = -\frac{C}{A} - \frac{Ce_c}{I} y_{top}$$

$$= -\frac{1620 \times 10^3}{375 \times 10^3} - \frac{1620 \times 10^3 \times 40}{1.758 \times 10^{10}} \times 375$$

$$= -4.32 - 1.38$$

$$= -5.7 \text{ N/mm}^2$$

Bottom fibre stress

$$(f_c)_b = -\frac{C}{A} + \frac{Ce_c}{I} y_{bot}$$

$$= -\frac{1620 \times 10^3}{375 \times 10^3} + \frac{1620 \times 10^3 \times 40}{1.758 \times 10^{10}} \times 375$$

$$= -4.32 + 1.38$$

$$= -2.9 \text{ N/mm}^2$$

c) Load balancing method

Effective upward load, w_{up} $= 8Pe / L^2$

$= 8 \times 1620 \times 10^3 \times 145 / 7300^2$

$= 35.3$ kN/m

Residual load w_{res} $= 45 - 35.3$

$= 9.7$ kN/m

Residual bending moment M_{res} $= 9.7 \times 7.32 / 8$

$= 64.6$ kNm

Residual bending stress $(f_c)_{res}$ $= 64.6 \times 10^6 \times 375 / 1.758 \times 10^{10}$

$= 1.38 \text{ N/mm}^2$

Total top fibre stress $(f_c)_t$ $= -P/A - (f_c)_{res}$

$= -4.32 - 1.38$

$= -5.7 \text{ N/mm}^2$

Total bottom fibre stress $(f_c)_b$ $= -P/A + (f_c)_{res}$

$= -4.32 + 1.38$

$= -2.9 \text{ N/mm}^2$

The resultant stress distribution at mid-span is shown below.