Module 9 : Infinite Series, Tests of Convergence, Absolute and Conditional Convergence, Taylor and Maclaurin Series

Lecture 27 : Series of functions [Section 27.1]

Objectives

In this section you will learn the following :

- Definition of power series.
- Radius of Convergence of power series.
- Differentiating and integrating power series.

27.1 Power Series

27.1.1 Definition:

(i) A series of the form

$$\sum_{n=1}^{\infty} a_n (x-c)^n = a_0 + a_1 (x-c) + a_2 (x-c)^2 + \ldots + a_n (x-c)^n + \ldots$$

is called a **power series** in the variable x centered at $c \in \mathbb{R}$, where $a_n \in \mathbb{R}$ for all n .

(ii) A power series $\sum_{n=1}^{\infty} a_n (x-c)^n$ is said to **converge** for a particular value $x = x_0$ if

the series $\sum\limits_{n=1}^{\infty} a_n (x_0 - c)$ is convergent

(iii) The set of all $x \in \mathbb{R}$ such that $\sum_{n=1}^{\infty} a_n (x-c)$ is convergent, is called the **domain of convergence** of the

power series.

27.1.2 Examples:

(i) Consider the power series

$$\sum_{n=0}^{\infty} x^n$$

centered at $c=0$. For a fixed value of x, this is a geometric series, and hence will be convergent for $|x|<1$, with sum $1/1-x$. Thus, we can write

$$\frac{1}{1-x}=\sum_{n=0}^{\infty} x^n, \text{ for } -1<x<1.$$

(ii)

The power series $\sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^n (x-3)^n$ is a power series centered at $c=3$. For every fixed value of x, this can be treated as a geometric series with common ratio $\left(-\frac{1}{3}\right)(x-3)$.

Thus, for a particular x, it will be convergent if $\left|\frac{x-3}{3}\right|<1$, i.e., $|x-3|<3$, i.e., $0<x<6$, and its sum is

$$\frac{1}{1-\frac{x-3}{3}}=\frac{3}{x}.$$

Hence,

$$\frac{3}{x}=\sum_{n=1}^{\infty}\left(-\frac{1}{3}\right)^n (x-3)^n, \text{ for } 0<x<6.$$

(iii) Consider the series

$$\sum_{n=1}^{\infty} (-1)^{n-1}\frac{(x-2)^n}{n}$$

This is a power series centered at $c=2$. For its convergence, let us apply the limit ratio test. Since

$$\left|\frac{(-1)^n (x-2)^{n+1}}{(-1)^{n-1} (n+1)}\frac{n}{(x-2)^n}\right|=\frac{n}{n+1}|x-2|\to|x-2|$$

the series is convergent absolutely for $|x-2|<1$, i.e., $1<x<3$, and is divergent for other values of x.

For $x=1$, the series is $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(-1)^n}{n}=\sum_{n=1}^{\infty}\frac{(-1)^{2n-1}}{n}$,

which is a divergent series. Also for $x=3$, it is the alternating harmonic series. Thus, the given power series is convergent with domain of convergence being the interval $(1,3]$.

The domain of convergence if a power series is given by the following theorem.

27.1.3 Theorem :

For a power series

$$\sum_{n=1}^{\infty} a_n (x-c)^n$$

precisely one of the following is true.

(i) *The series converges only for* $x = c$.

(ii) *There exists a real member* $R > 0$ *such that the series converges absolutely for* x *with* $|x-c| < R$ *, and diverges for* x *with* $|x-c| > R$

(iii) *The series converges absolutely for all* x .

27.1.4 Definition :

The **radius** of **convergence** R of a power series $\sum_{n=1}^{\infty} a_n (x-c)^n$, is defined to be number

(i) $R = 0$ if the series is divergent for all $x \neq c$.

(ii) $R = +\infty$ if, the series is absolutely convergent for all x .

(iii) R , the positive member such that the series diverges a for all x such that $|x-c| > R$ and the series converges

absolutely for all x such that $|x-c| < R$. The interval $I \subseteq \mathbb{R}$ such that the series converges is convergent for all $x \in I$ is called the **interval** of **convergence** .

27.1.5 Remark :

Note that the interval of convergence is either a singleton set, or a finite interval or the whole real line. In case it is a finite interval, the series may or may not converge at the and points of this interval. At all interior points of this interval, the series is absolutely convergent.

27.1.6 Example :

Consider the power series

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2}.$$

To find the value of x for which the series will be convergent, we apply the ratio test. Since

$\lim_{n\to\infty} \left| \dfrac{\frac{(x-2)^{n+1}}{(n+1)^2}}{\frac{(x-2)^n}{n^2}} \right| = \lim_{n\to\infty} |x-2| \left(\dfrac{n}{n+1} \right)^2 = |x-2|$, the series is absolutely convergent for x with $|x-2| < 1$,

i.e.,

absolutely convergent for $1 < x < 3$.

And the series is divergent if $x < 1$ or $x > 3$.

For $x = 1$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$,

which is absolutely convergent. Also for $x = 3$, the series is convergent. Hence, the series has radius of convergence $R = 1$, with interval of convergence $I = [1, 3]$.

For a power series

$$\sum_{n=1}^{\infty} a_n (x - x_0)^n$$

if I is the interval of convergence, then for every $x \in I$, let

$$f(x) := \sum_{n=1}^{\infty} a_n (x - x_0)^n, x \in I \cdot$$

Then

$$f : I \to \mathbb{R}$$

is a function on the interval I. The properties of this function are given by in the next theorems, which we assume without proof.

27.1.7 Theorem (Differentiation of power series) :

Let a power series

$$\sum_{n=1}^{\infty} a_n (x - c)^n$$

have non-zero radius of convergence R *and*

$$f(x) := \sum_{n=1}^{\infty} a_n (x - c)^n, x \in (c - R, c + R).$$

Then, the following holds:

The function f is differentiable on the interval $(c - R, c + R)$. Further the series $\sum_{n=1}^{\infty} \frac{d}{dx}(a_n (x - c)^n)$

also has radius of convergence R, and $f`(x) = \sum_{n=1}^{\infty} \frac{d}{dx}(a_n (x - c)^n), x \in (c - R, c + R)$.

The function f has derivatives of all orders and $f^{(k)}(x) = \sum_{n=1}^{\infty} \frac{d^k}{dx^k}(a_n (x - c)^n), x \in (c - R, c + R)$.

27.1.8 Theorem (Integration of power series):

Let

$$\sum_{n=1}^{\infty} a_n(x-c)^n$$

be a power series with non-zero radius of convergence R *and let*

$$f(x) := \sum_{n=1}^{\infty} a_n(x-c)^n, x \in (c-R, c+R).$$

Then

(i) The function f has an anti derivative $F(x)$ given by

$$F(x) = \int f(x)dx := \sum_{n=0}^{\infty} \frac{a_n(x-c)^{n+1}}{n+1} + C,$$

where C *is an arbitrary constant, and the series on the right hand side has radius of convergence* R.

(ii) *For* $[\alpha, \beta] \subset (c-R, c+R)$,

$$\int_{\alpha}^{\beta} f(x)dx = \sum_{n=1}^{\infty} \left[\int_{\alpha}^{\beta} a_n(x-c)^n dx \right],$$

where the series on the right hand side is absolutely convergent.

27.1.9 Example :

(i) Consider the power series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

By the ratio test, for every x

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0.$$

Hence, the series is absolutely convergent for every x. Let

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}, x \in \mathbb{R}.$$

Then, f is differentiable by theorem 27.1.7, and

$$f'(x) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(\frac{x^n}{n!}\right)$$

$$= \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

$$= f(x).$$

(ii) Consider the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{n}.$$

The power series is absolutely convergent (by ratio test) for

$|x|<1$

It is divergent for $x=1$ and convergent for $x=-1$. Thus

$$f(x)=\sum_{n=1}^{\infty} \frac{x^n}{n}, x \in [-1,1)$$

is defined. Since the serie

$$\sum_{n=1}^{\infty} \frac{d}{dx}\left(\frac{x^n}{n}\right)=\sum_{n=1}^{\infty} x^{n-1}$$

is convergent for $|x|<1$ and divergent for $|x|>1$, we have

$$f'(x)=\sum_{n=1}^{\infty} x^{n-1}, x \in (-1,1).$$

The series $\sum_{n=1}^{\infty} \int \frac{x^n}{n} dx = \sum_{n=1}^{\infty} \int \frac{x^{n+1}}{n(n+1)} dx$ is convergent, by the ratio test, for $|x|<1$ and divergent for $|x|>1$.. It is also convergent for $|x|=1$. Hence, it is convergent for $|x|\le 1$ and

$$\int f(x)dx=\sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)}, |x|\le 1.$$

Practice Exercises:

1. Find the radius of convergence of the power series:

(i) $\sum_{n=0}^{\infty} \frac{(-x)^n}{n+1}$.

(ii) $\sum_{n=0}^{\infty} \frac{(2x)^{2n}}{(2n)!}$.

(iii) $\sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$.

(iv) $\sum_{n=0}^{\infty} \frac{(x-1)^n}{10^n}$.

(v) $\sum_{n=0}^{\infty} (nx)^n$.

Answers

2. Find the interval of convergence of the following power series:

(i) $\sum_{n=0}^{\infty} \frac{(-x)^n}{n}$.

(ii) $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n3^n}$.

(iii) $\sum_{n=0}^{\infty} \frac{\sqrt{n}(x)^n}{3^n}$.

(iv) $\sum_{n=0}^{\infty} \frac{(4-3)^n}{n^{5/2}}$.

Answers

3. For the following power series, find the interval of convergence. If $f(x)$ is the function represented by it in the

respective interval of convergence, find $f'(x)$ and $\int f(x)ds$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-2)^{n+1}}{n+1}$$

Answer

4. Consider the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \text{ and } \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

Show that both the series have same interval of convergence. Find the relation between the functions represented by these series.

Answer

5. **Bessel Function of order zero:**

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(k!)^2}$$

Prove the following:

(i) The series converges for all x (use ratio test)

(ii) Let $J_0(x)$ denote the sum of this series. Show that J_0 satisfies the differential equation

$$x^2 J''_0(x) + xJ'_0(x) + x^2 J_0 = 0.$$

6. **Bessel functions of order one:**

Consider the power series

$$x\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n+1} n!\,(n+1)!}.$$

(i) Show that the series convergence for all x.

(ii) If $J_1(x)$ denote the sum of this series, show that

$$x^2 J''_1(x) + x\int_1^1 (x) + (x^2-1)\, J_1(x) = 0.$$

(iii) Show that

$J'_0(x) = -J_1(x).$

Recap

In this section you have learnt the following

- Definition of power series.
- Radius of Convergence of power series.
- Differentiating and integrating power series.

Series, Tests of Convergence, Absolute and Conditional Convergence, Taylor and Maclaurin Series

Lecture 27 : Taylor and Maclaurin series [Section 27.2]

Objectives

In this section you will learn the following :

- Taylor series expansion for functions.
- Maclaurin series expansion functions.

27.2 Taylor Series and Maclaurin series

In section we saw that a function can be approximated by a polynomial of degree n depending upon its order of smoothness. If the error terms converge to zero, we set a special power series expansion for f .

27.2.1 Definition:

Let $f : I = (a-\delta, a+\delta) \rightarrow \mathbb{R}$ be a function which has derivative of all order in I . Then the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{(n)!}(x-a)^n,$$

is called the **Taylor series** for f at $x=a$. We say f has **Taylor series expansion** at $x=a$, if its Taylor series is convergent for $x \in I$ and its sum is $f(x)$. For $a=0$, the Taylor series for f is called the **Maaculurin Series** for f at $x=0$.

27.2.2 Examples:

(i) For the function $f(x)=\dfrac{1}{x}$, $x \neq 0$, its derivatives of all order exist in domain

$$I=(-\infty,0)\cup(0,\infty).$$

For a = 1, since

$$f^{(n)}(x)=(-1)^n n!\, x^{-(n+1)},$$

we have

$$f^{(n)}(1)=(-1)^n n!, \forall n \geq 1 \quad .$$

Thus

$$\sum_{n=0}^{\infty}(-1)^n (x-1)^n$$

is its Taylor series at $x=1$. Since it is a geometric series, it will be convergent if

$$|x-1|<1 \; i.e., \; 0<x<2 .$$

Further, its sum is

$$\frac{1}{1+(x-1)}=\frac{1}{x}.$$

Hence, $f(x)=1/x$ has Taylor series expansion

$$\frac{1}{x}=\sum_{n=0}^{\infty}(-1)^n (x-1)^n \; , 0<x<2.$$

(ii) Consider the function $f(x)=e^x$, $x \in \mathbb{R}$ since

$$f^{(n)}(x)=e^x \quad \forall n \geq 1,$$

The Taylor (Maclaurin) Series of f at $a=0$, is given by

$$f(0)+f'(0)x+\frac{f''(0)}{2!}x+\ldots=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

By the ratio test, series converges for all x, but we do not know its sum.

27.2.3 Theorem (Convergence of Taylor Series):

Let I be an open interval and $f: I \rightarrow \mathbb{R}$ be a function having derivatives of all order in I. For $a, x \in I$, for every $n \geq 1$, there exists a point C_n between a and x such that

$$f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k+\frac{f^{(n+1)}(C_n)}{(n+1)!}(x-a)^{n+1}.$$

Further, the Taylor series of f at $x = a$ *converges to* $f(x)$ *if*

$$R_n(x) := \frac{f^{(n+1)}(C)}{(n+1)!}(x-a)^{n+1} \to 0 \text{ and } n \to \infty.$$

27.2.3Theorem (Convergence of Taylor Series):

Let I *be an open interval and* $f : I \to \mathbb{R}$ *be a function having derivatives of all order in* I*. For* $a, x \in I$, *for every* $n \geq 1$ *there exists a point* C_n *between a and* x *such that*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(C_n)}{(n+1)!}(x-a)^{n+1}.$$

Further, the Taylor series of f at $x = a$ *converges to* $f(x)$ *if*

$$R_n(x) := \frac{f^{(n+1)}(C)}{(n+1)!}(x-a)^{n+1} \to 0 \text{ and } n \to \infty.$$

Proof

Follows from theorem 14.1.1 we have already seen some examples of Taylor series expansion in section 14.1 we give some more examples.

27.2.4 Examples:

(i) As in example 27.2.2(*ii*), for $f(x) = e^x$, with $a = 0$,

$$R_n(x) := \frac{e^C}{(n+1)!}(x)^{n+1}$$, for some c between 0 and x.

For $x < 0$, $e^C < 1$ as c is between x and 0. For $x > 0$, since e^x is monotonically increasing, $e^C < e^x$. They

$$|R_n(x)| \leq \begin{cases} \dfrac{|x|^{n+1}}{(n+1)!} & \text{for } x < 0 \\[2ex] \dfrac{e^x\,|x|^{n+1}}{(n+1)!} & \text{for } x > 0 \end{cases}$$

Since,

$$\frac{|x|^{n+1}}{(n+1)!} \to 0 \text{ as } n \to \infty$$

For every *x* fixed, we have

$$\lim_{n \to \infty} R_n(x) = 0.$$

Hence, the Taylor series of $f(x) = e^x$ indeed converges to the function $f(x)$, i.e.,

$$e^x = \sum_{k=0}^{\infty} \frac{x^n}{n!}$$

(ii) For the function

$$f(x) = \cos\ x,\ x \in \mathbb{R},$$

since

$$f^n(0) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^k & n = 2k \end{cases}$$

and $\left|f^n(x)\right| \le 1$ for all x, we have

$$\left|R_n(x)\right| = \left|\frac{f^{(n+1)(C)}}{(n+1)!}(x)^{n+1}\right| \le \frac{|x|^{n+1}}{(n+1)!} \to 0 \text{ and } n \to \infty$$

Hence, Taylor series of $f(x)$ = cos x is convergent to $f(x)$, and

$$\cos\ x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

Similarly, one can show

$$\sin\ x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

27.2.5 Note:

(i) Suppose, a power series

$$\sum_{n=0}^{\infty} a_n (x-a)^n$$

is convergent is an open interval *I* around a point a $\mathbb{R}$, and

$$f(x) := \sum_{n=0}^{\infty} a_n (x-a)^n \quad x \in I \qquad \text{----------(*)}$$

A natural question arises, in the power series above the Taylor series expansion of $f(x)$ The answer is yes. In fact if (*) holds, then the power series has nonzero radius of convergence and hence by theorem 27.1.7, series can be differentiated term by term, giving

$$f^n(x) = n!a_n + (n-1)!a_{n-1}(x-a) + \ldots + \ldots.$$

Thus

$$f^n(a) = n!a_n; \quad n \ge 1$$

(ii) In view of (i) above, if $f(x)$ is expressed as sum of a power series, then it must be Taylor series of $f(x)$.

Thus, technique of previous section can be used to find Taylor series expansions.

27.2.6 Examples :

(i) From the convergence of geometric series, we know

$$\frac{1}{1-x} = 1 + x + \ldots + x^n + \ldots, \quad |x| < 1,$$

Thus, this is the Maclaurin series for $f(x) = \dfrac{1}{1-x}$. If we change the variable from x to $-x^2$, we get

$$\frac{1}{1-x^2} = 1 - x^2 + x^4 - x^6 + \ldots, \quad |x| < 1,$$

the Maclaurin Series expansion for $f(x) = \dfrac{1}{1 - x^2}$. (Note that to find Maclaurin Series directly requires some tedious derivative computations). Now for $x \in (-1,1)$, using theorem 27.1.8, we have

$$\tan^{-1} x + c = \int \frac{1}{1+x^2}\, dx = \int x^2 dx + \int x^4 dx \ldots$$

$$= c + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

.

Since for $x = 0$, $\tan^{-1} x = 0$, we have $c = 0$. Hence

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots, |x| < 1.$$

In fact, using the alternative series test, it is easy to see that the above holds for $x = \pm 1$ also.

27.2.7 Algebraic operations on power series:

Suppose power series

$$\sum_{n=1}^{\infty} a_n (x-a)^n \text{ and } \sum_{n=1}^{\infty} b_n (x-a)^n$$

are both absolutely convergent to $f(x)$ and $g(x)$ respectively $|x| < R$. Then it can be shown that the following series are absolutely convergent to $|x| < R$.

(i) $f(kx) = \sum_{n=0}^{\infty} a_n (kx - a)^n$.

(ii) $f(x^N) = \sum_{n=0}^{\infty} a_n (x^N - a)^n$.

(iii) $f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n)(x-a)^n$.

(iv) $f(x) g(x) = \sum_{n=0}^{\infty} C_n (x-a)^n$.

where

$$C_n = \sum_{n=0}^{\infty} a_n b_n - k \;, \; n \geq 1.$$

This can be used to write Taylor series expansions from known expression.

27.2.8 Examples:

(i) Let us find Maclaurin series of the function

$$f(x)=\frac{3x-1}{x^2-1} \quad , \; x\neq \pm 1.$$

Since

$$\frac{3x-1}{x^2-1}=\frac{2}{x+1}+\frac{1}{x-1}.$$

and

$$\frac{1}{x-1}=-\sum_{n=0}^{\infty} x^n \quad , \text{ for } |x|<1,$$

$$\frac{2}{x+1}=2\sum_{n=0}^{\infty}(-1)^n x^n, \quad \text{for } |x|<1$$

We have

$$\frac{3x-1}{x^2-1}=\sum_{n=0}^{\infty}\left(2(-1)^n-1\right)x^n \quad , \quad |x|<1.$$

Thus

$$\frac{3x-1}{x^2-1}=1-3x+x^2+x^2-3x^3+\ldots \, , \; |x|<1,$$

is the Maclaurin series of $f(x)$.

(ii) Let

$$f(x)=e^{-x^2}\tan^{-1}x.$$

Since

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!} \quad , \quad \text{for all x},$$

we have

$$e^{-x^2}=\sum_{n=0}^{\infty}\frac{\left(-x^2\right)^n}{n!} \; , \quad \text{for all } x \quad .$$

Also

$$\tan^{-1}x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1} \quad , \quad \text{for } |x|<1.$$

We have

$$f(x)=e^{-x^2}\tan^{-1}x=\left(\sum_{n=0}^{\infty}\frac{\left(-x^2\right)^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}\right), \text{for } |x|<1$$

$$= x-\frac{4}{3}x^3+\frac{31}{30}x^5\ldots, \quad |x|<1,$$

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Practice Exercises:

(1) Using definition, find the Taylor series of $f(x)$ around the point C:

(i) $f(x)=ln(x), C=1$.

(ii) $f(x) = \cos(x), \ x = \pi/4$.

Answer

(2) Making appropriate substitutions in a known Maclaurin series, find the Maclaurin series of the following along

with its radius of convergence:

(i) $\sin(2x)$

(ii) e^{x^2}

(iii) $2\cos^2 x$

Answer

(3) Maclaurin series for the following.

(i) $\sqrt[3]{1+x^2}$

(ii) $ln(x+\sqrt{x^2+1})$

(iii) $\sqrt{1+x}\,ln(1+x)$

(iv) $\sinh^{-1} x$

Answer

(4) Using Maclaurin series for standard functions and suitable operations, write Maclaurin series for the following :

(i) $x^2 \cos \pi x$

(ii) $e^x \sin x$

(iii) $\dfrac{-2}{x^2-1}$

(iv) $ln(1-x^2)$

Answer

(5) Binomial series:

Write Maclaurine series for the function

$f(x) = (1+x)^m$

Where *m* is a real member. Using ratio test, show that the Maclaurin series for $f(x)$ is convergent for $|x| < 1$. This series is also called Binomial series for $f(x)$. Using this series, find the Maclaurin series for $f(x) = (1+x)^{-1/2}$.

Answer

Recap

In this section you have learnt the following

- Taylor series expansion for functions.

- Maclaurin series expansion functions.