8 Dispersion and Propagation of Wave Packets

Keywords: Dispersion, group velocity, wave packet, amplitude equation

8.1 Dispersion

Consider a dispersion relation of the form $\omega = \omega(k)$. Superposing harmonic waves with Fourier amplitude $F(k)$ in such a medium gives

$$f(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k)e^{i(kt - \omega(k)t)} \, dk.$$  \hfill (8.1)

It is intuitively evident that, as time progresses, the wave-form is going to get modified, a phenomenon known as dispersion. This occurs because now the Fourier amplitude of the wave-form may considered to be $F(k)e^{-i\omega(k)t}$, which is time-varying. However, in a very special case (or medium), if $\omega(k)/k = \beta$ (constant), then (8.1) may be simplified as

$$f(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k)e^{ik(x - \beta t)} \, dk = f(x - \beta t).$$  \hfill (8.2)
Thus, in this special situation, the wave-form propagates without distortion at the speed $\beta = \omega/k$. Such a medium is known as a non-dispersive medium. When a wave-form travels through a dispersive medium it will distort since the phase speeds of the harmonic components constituting the wave-form are all different. Hence, we require a different definition of speed of the wave-form in a dispersive medium, which is discussed next.

### 8.2 Group Velocity

Consider the Fourier representation of a traveling wave in a dispersive medium of the form

$$f(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) e^{i(kx - \omega(k)t)} \, dk. \quad (8.3)$$

We will consider a narrow-band wave with a spectrum distribution similar to that shown in Fig. 8.1(a). The corresponding wave-form is an amplitude
modulated wave like the one shown in Fig. 8.1(b), and is termed as a wave-packet. Such a wave-packet has a functional representation of the form

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} F(k)e^{ikx} \, dk = f_0(x)e^{ik_0x},
\]

(8.4)

where \(f_0(x)\) represents the modulating envelop, as shown in Fig. 8.1(b). It is of interest to determine the velocity of the profile of the wave-packet since the energy of the packet is localized in the high amplitude region.

Let us take the central wave number as \(k_0\), and write \(k = k_0 + \xi\), where \(\xi \in [-\infty, \infty]\). Then, one can easily show from (8.4) that

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} F(k_0 + \xi)e^{i\xi x} \, d\xi = f_0(x).
\]

(8.5)

Using this, one can now rewrite (8.3) as

\[
f(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k_0 + \xi)e^{i[k_0x+\xi x-\omega(k_0)\xi]t} \, d\xi,
\]

or

\[
f(x, t) \approx e^{i(k_0x-\omega(k_0)t)} \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k_0 + \xi)e^{i\xi[x-\omega'(k_0)t]} \, d\xi,
\]

or

\[
f(x, t) = f_0[x - \omega'(k_0)t] e^{i(k_0x-\omega(k_0)t)} \quad \text{(using (8.5))},
\]

or

\[
f(x, t) = f_0[x - c_G t] e^{ik_0(x-c_P t)},
\]

(8.6)
where \( c_P = \omega(k_0)/k_0 \), and

\[
c_G := \omega'(k_0) = \left. \frac{d\omega(k)}{dk} \right|_{k=k_0},
\]

is defined as the group velocity of the wave. It is evident from (8.6) that the amplitude envelope \( f_0[\cdot] \) of a wave-packet travels at the group velocity \( c_G \), while the wave itself travels at the phase velocity given by \( c_P = \omega(k_0)/k_0 \). It should be noted that the solution (8.6) is valid only for a small time due to the approximation involved in deriving it. The concepts of \( c_P \) and \( c_G \) in a dispersive medium are graphically explained in Fig. 8.2, where the slopes of the dashed lines yield the respective values.

\[ \omega \]
\[ k \]

\[ c_G = \frac{d\omega}{dk} \]
\[ D(\omega, k) = 0 \]

\[ c_P = \frac{\omega}{k} \]

**Fig. 8.2:** The concept of phase and group velocities in a dispersive medium

### 8.3 Evolution of a Gaussian Wave Packet

Consider the propagation of a narrow-banded Gaussian wave packet represented in the Fourier domain as \( F(k) = e^{-\alpha^2(k-k_0)^2} \). We are interested in evolution of the packet shape as it propagates in a dispersive medium. We
can improve upon the calculation of the Fourier integral presented in the previous section by expanding (8.3) as

\[
f(x, t) \approx e^{i(k_0 x - \omega(k_0) t)} \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k_0 + \xi)e^{i[x\xi - \omega'(k_0) t\xi - \omega''(k_0) t\xi^2/2]} d\xi,
\]

\[
= e^{i(k_0 x - \omega(k_0) t)} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-[\beta^2\xi^2 + 2\beta\gamma\xi]} d\xi,
\]

where \(\beta = \alpha^2 - i\omega''(k_0)t/2\), and \(2\beta\gamma = i(\omega'(k_0)t - x)\). Simplifying the above expression yields

\[
f(x, t) \approx e^{i(k_0 x - \omega(k_0) t)} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{[-(\beta\xi + \gamma)^2 + \gamma^2]} d\xi
\]

\[
= \frac{1}{\sqrt{4\pi(\alpha^2 - i\omega''(k_0)t/2)}} e^{-\frac{(\omega'(k_0)t - x)^2}{4(\alpha^2 - i\omega''(k_0)t/2)}} e^{i(k_0 x - \omega(k_0) t)}
\]

(8.7)

This expression is composed of a harmonic wave of wave number \(k_0\) and frequency \(\omega(k_0)\), an envelop which is traveling at a speed \(\omega'(k_0)\) (group velocity), and a temporal attenuating factor which falls as \(\sim 1/\sqrt{t}\). It may be noted that the evolution of the amplitude \(A(x, t)\) of the harmonic wave \(e^{i(k_0 x - \omega(k_0) t)}\) in (8.7) is governed by the amplitude equation (Schrödinger equation)

\[
A_t - \omega'(k_0) A_x - \frac{i\omega''(k_0)}{2} A_{xx} = 0
\]

which has a diffusive character. Thus, the spread of the packet will increase while its amplitude will fall as it propagates.
For a string supported on a flexible foundation of stiffness density $\kappa$, the dispersion relation is given by $\omega^2 = c^2 k^2 + \kappa/\rho A$. The propagation of a Gaussian wave packet with $\alpha = 4$ and $k_0 = 1$ is shown in Fig. 8.3. The values of the constants were taken as $c = \sqrt{200}$ m/s and $\kappa/\rho A = 500$ rad$^2$/s$^2$. 