Vibrations of Structures

Module III: Vibrations of Beams

Lesson 20: Beam Models - II

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Keywords: Beam models, Shear deformation, Timoshenko beam, Variational formulation, Higher order beam models
1 The Timoshenko Beam

Timoshenko beam model takes into account the shear deformation of a beam cross-section.

The Newtonian Formulation

The Timoshenko beam requires two field variables: $\psi(x, t)$ for flexure and $\theta(x, t)$ for shear, as shown in Fig. 1. One can write

$$w_{xx}(x, t) = \psi(x, t) + \theta(x, t)$$

(1)
where \( w(x, t) \) represents the displacement field of a neutral fiber of the beam.

The relation (1) allows us to choose of \( w(x, t) \) and \( \psi(x, t) \) as the two field variables.

Since the longitudinal strain \( \epsilon_x \) in the beam is produced only from bending, one can write

\[
\epsilon_x = \frac{(R - z) \frac{d\psi}{dx} - R \frac{d\psi}{dx}}{dx} = -z \psi_{,x}. \tag{2}
\]

Using Hooke’s law, the longitudinal stress is obtained as \( \sigma_x = -Ez\psi_{,x} \), and the bending moment can be expressed as

\[
M = - \int_{-h/2}^{h/2} \sigma_x z \, dA = EI \psi_{,x}. \tag{3}
\]

Similarly, the net shear force acting at a section can be written as

\[
V = G A_s \theta = G A_s (w_{,x} - \psi),
\]

where \( A_s = A/\kappa \), \( A \) is the area of cross-section of the beam, and \( \kappa \) is known as the shear correction factor. This factor is introduced to take care of the non-uniformity in the shear force across the section. For a rectangular section \( \kappa \approx 1.20 \), for a circular section \( \kappa \approx 1.11 \), and for an I-section \( \kappa \approx 2.24 \). Newton’s second law for the transverse motion yields

\[
(\rho A \, dx)w_{,tt} = V(x + \, dx) - V(x) \quad \Rightarrow \quad \rho A w_{,tt} = V_{,x},
\]

or \( \rho A w_{,tt} = [G A_s (w_{,x} - \psi)]_{,x} \). \tag{4}
The equation of rotational dynamics can be written as

\((\rho I \, d\psi)_{,tt} = V \frac{dx}{2} + (V + dV) \frac{dx}{2} + (M + dM) - M\)

or \(\rho I \psi_{,tt} = V + \frac{dM}{dx}\),

or \(\rho I \psi_{,tt} = GA_s(w_{,x} - \psi) + [EI \psi_{,x}]_{,x}\). \hspace{1cm} (5)

The two differential equations (4) and (5) in \(w(x, t)\) and \(\psi(x, t)\) represent the dynamics of a Timoshenko beam.

In the case of a uniform beam, we have the simplification

\(\rho A w_{,tt} = GA_s(w_{,xx} - \psi_{,x})\), \hspace{1cm} (6)

and \(\rho I \psi_{,tt} = GA_s(w_{,x} - \psi) + EI \psi_{,xx}\). \hspace{1cm} (7)

Differentiating (7) once with respect to \(x\) we have

\(\rho I \psi_{,x,tt} = GA_s(w_{,xx} - \psi_{,x}) + EI \psi_{,xxx}\). \hspace{1cm} (8)

Solving for \(\psi_{,x}\) from (6) and substituting in (8) yields on simplification

\(\frac{\rho I}{GA_s} w_{,tttt} + w_{,tt} - \left( \frac{I}{A} + \frac{EI}{GA_s} \right) w_{,ttxx} + \frac{EI}{\rho A} w_{,xxxx} = 0\). \hspace{1cm} (9)

With the definitions \(c_L = \sqrt{E/\rho}\) (longitudinal wave speed), \(c_S = \sqrt{G/\rho}\) (shear wave speed), and \(r_g = \sqrt{I/A}\) (radius of gyration), the equation of transverse motion of a uniform Timoshenko beam can be written as

\(\left( c_L^2 \frac{\partial^2 \psi_{,x}}{\partial x^2} - \frac{\partial^2 \psi_{,tt}}{\partial t^2} \right) \left( \frac{c_S^2}{\kappa} \frac{\partial^2 \psi_{,xx}}{\partial x^2} - \frac{\partial^2 \psi_{,tt}}{\partial t^2} \right) w + \frac{c_S^2}{\kappa r_g^2} \frac{\partial^2 w}{\partial t^2} = 0\). \hspace{1cm} (10)
The Variational Formulation

The total kinetic energy density of a beam element consists of the translational and rotational kinetic energy densities, and is given by

\[ \hat{T} = \frac{1}{2} \rho A w^2_t + \frac{1}{2} \rho I \psi^2_t. \]  

(11)

The potential energy density can be written as

\[ \hat{V} = \frac{1}{2} EI \psi^2_{,x} + \frac{1}{2} GA_s \theta^2 = \frac{1}{2} EI \psi^2_{,x} + \frac{1}{2} GA_s (w_{,x} - \psi)^2 \]  

(12)

Hamilton’s variational principle for the dynamics of the beam yields

\[ \delta \int_{t_1}^{t_2} \int_0^l (\hat{T} - \hat{V}) \, dx \, dt = 0, \]

\[ \Rightarrow \int_{t_1}^{t_2} \int_0^l \left[ \rho A w_{,t} \delta w_{,t} + \rho I \psi_{,t} \delta \psi_{,t} - \right. \]

\[ EI \psi_{,x} \delta \psi_{,x} - GA_s (w_{,x} - \psi)(\delta w_{,x} - \delta \psi) \]  

\[ dx \, dt = 0. \]  

(13)

Integrating by parts and rearranging, we have

\[ \int_0^l \left[ \rho A w_{,t} \delta w + \rho I \psi_{,t} \delta \psi \right] \bigg|_{t_1}^{t_2} \, dx + \int_{t_1}^{t_2} \left[ -EI \psi_{,x} \delta \psi - GA_s (w_{,x} - \psi) \delta w \right] \bigg|_0^l \, dt \]

\[ + \int_{t_1}^{t_2} \int_0^l \left[ (-\rho A w_{,tt} + [GA_s (w_{,x} - \psi)],_x) \delta w + \right. \]

\[ (-\rho I \psi_{,tt} + (EI \psi_{,x},_x + GA_s (w_{,x} - \psi)) \delta \psi \]  

\[ dx \, dt = 0. \]  

(14)

The first integral above vanishes from the statement of the variational principle. The integrand in the double integral yields the two equations of motion (4) and (5). A set of possible boundary conditions is obtained from the
integration of the second integral in (14) as

\[ [EI\psi_x](0, t) = 0 \quad \text{or} \quad \psi(0, t) = 0, \]
and \[ [EI\psi_x](l, t) = 0 \quad \text{or} \quad \psi(l, t) = 0, \]
and \[ [GA_s(w_x - \psi)](0, t) = 0 \quad \text{or} \quad w(0, t) = 0, \]
and \[ [GA_s(w_x - \psi)](l, t) = 0 \quad \text{or} \quad w(l, t) = 0. \]

2 Higher Order Beam Theories

One may extend and generalize the above discussed standard beam models for improved accuracy (though with higher complexity) as follows. Consider the expansion of axial and transverse deformations of any material point of the beam in terms of the transverse coordinate \( z \) measured from the middle plane of the beam as, respectively,

\[ U(x, z, t) = \psi_0(x, t) + z\psi_1(x, t) + z^2\psi_2(x, t) + \ldots \] (15)
\[ W(x, z, t) = w_0(x, t) + zw_1(x, t) + z^2w_2(x, t) + \ldots \] (16)

where \( \psi_n(x, t) \) and \( w_n(x, t) \) are the field variables. It may be noted that \( \psi_0(x, t) \) introduces stretch of the middle plane of the beam. One may then compute the strain field using the definitions

\[ \epsilon_{xx} = U_{,x}, \quad \epsilon_{xz} = \frac{1}{2}(U_{,z} + W_{,x}), \quad \epsilon_{zz} = W_{,z}, \]

and the corresponding stress field can be obtained using Hooke’s law.
From this point, it is convenient to follow the variational formulation to derive the equations of motion of the beam. The kinetic energy of the beam can be calculated as

\[ T = \frac{1}{2} \int_0^l \int_A \rho (U_i^2 + W_i^2) \, dA \, dx. \]

The strain energy function may be expressed as

\[ V = \frac{1}{2} \int_0^l \int_A \left( \sigma_{xx} \epsilon_{xx} + 2 \sigma_{xz} \epsilon_{xz} + \sigma_{zz} \epsilon_{zz} \right) \, dA \, dx. \]

In the above energy expressions, one can carry-out the area integration as done before. Finally, following the variational procedure, one obtains the equations of motion for the field variables.

Consider the following two special cases.

(A) \( \psi_1(x, t) = -w_x(x, t) \) and \( w_0(x, t) = w(x, t) \):

We have

\[ U(x, z, t) = -zw_x(x, t), \quad \text{and} \quad W(x, z, t) = w(x, t). \]

Thus, \( \epsilon_{xx} = -zw_{xx} \) and \( \epsilon_{xz} = \epsilon_{zz} = 0 \), and using Hooke’s law, \( \sigma_{xx} = -zEw_{xx}. \) The kinetic energy and potential energy expressions are obtained as

\[ T = \frac{1}{2} \int_0^l (\rho I w_{xt}^2 + \rho A w_{t}^2) \, dx, \quad V = \frac{1}{2} \int_0^l EI w_{xx}^2 \, dx. \]

This then leads to the Rayleigh beam theory.
Figure 2: Visualization of the deformation kinematics in different beam theories (dashed line represents deformation under Euler-Bernoulli hypothesis)

(B) $\psi_1(x, t) = -\psi(x, t)$ and $w_0(x, t) = w(x, t)$:

Here,

$U(x, z, t) = -z\psi(x, t), \quad \text{and} \quad W(x, z, t) = w(x, t)$.

Hence, $\epsilon_{xx} = -z\psi_{,x}$ and $\epsilon_{xz} = 1/2(-\psi + w_{,x})$ and $\epsilon_{zz} = 0$. The stress field is obtained as $\sigma_{xx} = -zE\psi_{,x}$ and $\sigma_{xz} = 1/2G(w_{,x} - \psi)$. The kinetic energy and potential energy expressions are given by

$$T = \frac{1}{2} \int_0^l (\rho I \psi_x^2 + \rho A w_t^2) dx, \quad V = \frac{1}{2} \int_0^l [EI \psi_{,x}^2 + GA(w_{,x} - \psi)^2] dx.$$

This yields the Timoshenko beam theory.

The deformation kinematics of Euler-Bernoulli, Timoshenko and higher order beam theories are visualized in Fig. 2.