Stability of a Dynamic System

Dr. Bishakh Bhattacharya

Professor, Department of Mechanical Engineering

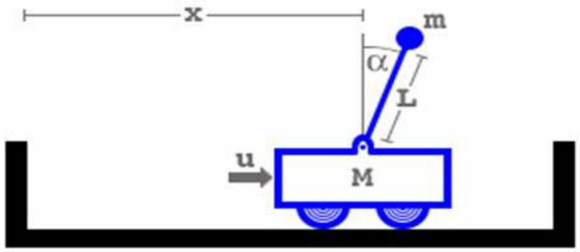
IIT Kanpur

This Lecture Contains

- Introduction
- Simple Test of Stability
- Routh Stability Criteria
- Special Cases
- An Assignment

Introduction

- **The response of a Dynamic System may become unbounded while subjected to a bounded input. Such systems are referred as unstable systems. One common example is an inverted pendulum on a rolling cart as shown below:**



- **A Control system could be designed such that by controlling the velocity of the rolling cart one can control the unstable response of the inverted pendulum.**

- **However, we need to first carry out a stability analysis of the system.**

How to test the stability of a system

- **A simple method to test the stability of a system is by checking the poles of the system transfer function.**
- **Consider a system which is represented by a generalized transfer function as follows:**

$$T(s)=\frac{N(s)}{D(s)}=\frac{c_0 s^m + c_1 s^{m-1} + \dots + c_{m-1} s + c_m}{a_0 s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n}$$

- **Now, equating the denominator polynomial D(s) to zero, one can obtain the characteristic equation for the system. The roots of this characteristic equations are the poles of the system.**
- **If you obtain one or more poles with positive real part then the system could be predicted to be an unstable system.**
- **However, it is often tedious to obtain the poles of a complex system before predicting stability condition of the system.**

Routh’s Test for Stability

For a characteristic polynomial D(s), the number of poles in the right-half plane may be determined without actually finding the roots by using the Routh Test.

The Routh array for the polynomial D(s) could be constructed as follows:

s^n	a_0	a_2	a_4	a_6 ...
s^{n-1}	a_1	a_3	a_5	a_7 ...
s^{n-2}	b_1	b_2	b_3	...
:	:	:	:	
s^0				

Routh's Theorem

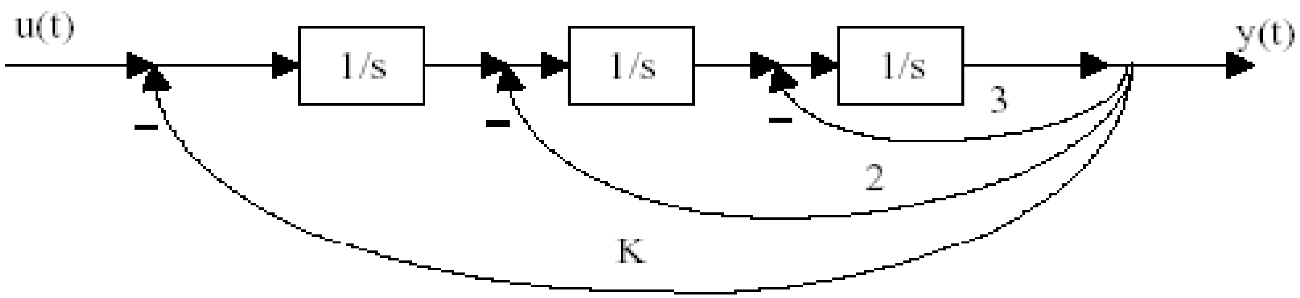
You may have observed that the first row of the Routh table consists of odd coefficients of D(s) starting from the first coefficient related to s^n . Again, the second row consists of the even coefficients starting from the second coefficient related to s^{n-1} .

The coefficients b_1 etc. For the third row could be computed as follows. The same pattern could be used for the subsequent rows.

$$b_1 = \frac{-\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix}}{a_1}, \quad b_2 = \frac{-\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix}}{a_1}, \quad b_3 = \frac{-\begin{vmatrix} a_0 & a_6 \\ a_1 & a_7 \end{vmatrix}}{a_1}.$$

Routh's Theorem: The number of roots of the characteristic polynomial D(s) in the right-half plane equals the number of sign changes in the first column of the Routh Table.

Stability as a function of a parameter



1. Use Mason's rule to find out the determinant of the system
2. Numerator of the determinant will be the characteristic polynomial
3. Use Routh's test and obtain the array
4. Note the conditions for stability analysing the left coloumn

Unusual Case: Left Column Zero

- Consider $D(s) = 3s^4 + 6s^3 + 2s^2 + 4s + 5$
- Note appearance of zero in the first column
- Rename the row as Row A
- Create row B from Row A by sliding the A row to left until you get a non-zero pivot
- The sign of the row is changed by $(-1)^n$ where n is the number of times this row is slided
- The new non-zero row is formed by adding A and B
- [Reference Benedir and Picinbond, IEEE Trans on Automatic Control, 1990]

Other approaches for zero left column

Alternate approaches:

Put a parameter, say ε instead of zero in the pivot, assuming it to be a very small positive number

Continue and find sign changes

OR

Write the polynomial in reverse order such that the roots of the reverse polynomial will be the reciprocal of the roots of the original polynomial and follow the same procedure

Assignment - Unusual Case: A zero row

Consider the polynomial D(s) as
$s^8 + 3s^7 + 10s^6 + 24s^5 + 48s^4 + 96s^3 + 128s^2 + 192s + 128$

Find out the stability of the system.

Hints: When you encounter a zero row, go back to the last row, construct the corresponding polynomial (even/odd).

Differentiate the polynomial and obtain the new non-zero row.

Special References for this lecture

- **Control Engineering and introductory course, Wilkie, Johnson and Katebi, PALGRAVE**
- ***Control Systems Engineering* – Norman S Nise, John Wiley & Sons**
- ***Modern Control Engineering* – K. Ogata, Prentice Hall**