NPTEL web course on Complex Analysis

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Module: 5: **Consequences of Complex Integration**
Lecture: 4: **Zeros and singularities**

Classification of Singularities

Classification of Singularities

Zeros of analytic function

Definition

A point α is called **zero** of order n of the analytic function $f(z)$ if we can write $f(z) = (z - \alpha)^n g(z)$ where $g(z)$ is analytic and $g(\alpha) \neq 0$.

Classification of Singularities

Zeros of analytic function

- The function $f(z) = (z - 2)$ has a 'zero' at $z = 2$.
- The function $\sin(\frac{1}{z})$ has 'zeros' at $z = \pm 1/n\pi$.
- There are no zeros of infinite order.

Classification of Singularities

Zeros of analytic function

Theorem

If $f(z)$ is analytic and not identically zero, then zeros of $f(z)$ are isolated.

Another version

Let $f(z)$ is analytic and has a zero, say z_0, then in the neighbourhood of z_0, $f(z)$ has no other zeros. Otherwise $f(z)$ is identically zero.

Proof

- Let $f(z)$ be analytic in D.
- Let $z_1, z_2, \ldots z_n$ be n zeros of $f(z)$ in D.
- Then $f(z) = (z - z_1)(z - z_2)\cdots(z - z_n)g(z)$ such that $g(z) \neq 0$ in D.
- Let $g(z_j) = a_m \neq 0$ for some fixed z_j.
- Since $g(z)$ is continous in D, $|g(z) - g(z_j)| < \epsilon$ whenever $|z - z_j| < \delta$.
- Choose $\epsilon = \dfrac{|a_m|}{2}$ implies $|g(z) - g(z_j)| < \dfrac{|a_m|}{2}$ for $|z - z_j| < \delta$.
- If $g(z) = 0$ in $|z - z_j| < \delta$, then $|a_m| < \dfrac{|a_m|}{2}$ which is a contradiction.

Definition

If the function $f(z)$ fails to be analytic at a point z_0, but analytic at some point in every neighbourhood of z_0, then z_0 is called the **singular point** of the function $f(z)$.

Classification of Singularities

Example

- For the function $f(z) = \dfrac{1}{z-\alpha}$, α is the singular point.
- The function $f(z) = |z|^2$ is nowhere analytic and hence has no isolated singular points. The function $f(z) = \log(z)$ has singularity at z=0.

Classification of Singularities

Definition

The Singularity at z_0 of a function $f(z)$ is said to be **isolated** if $\exists$ at least one neighborhood of z_0 in which $f(z)$ has no singularity other than z_0. Otherwise it is called the non-isolated Singularity.

Example

- Let $f(z) = \dfrac{1}{(z-1)(z-2)(z-3)}$, here $z = 1, 2, 3$ are the isolated singular points.
- The function $f(z) = \dfrac{z+3}{z^2(z^2+4)}$ has $z = 0$, $z = \pm 2i$ as isolated singularities.
- $z = 0$ is non-isolated singularity for the function $f(z) = \log z$, since the function fails to be analytic in the entire negative real axis.

Classification of Singularities

Definition

The singularity $z = z_0$ is called a **removable singularity** of $f(z)$ if $\lim_{z \to \alpha} (z - \alpha) f(z) = 0$. In other words, if $f(z)$ has a singularity at the point z_0 but the Principal part of the Laurent series expansion of $f(z)$ about z_0 has no terms then $z = z_0$ is removable singularity of $f(z)$.

Example

- The function $f(z) = \dfrac{\sin z}{z}$ has a removable singularity at $z = 0$.
- The function $f(z) = \dfrac{\exp(z) - 1}{z}$ has a removable singularity at $z = 0$.

Classification of Singularities

Definition

The point $z = z_0$ is called a **pole** of order 'm' if $f(z) = \dfrac{\phi(z)}{(z - z_0)^m}$ where $\phi(z)$ is analytic and $\phi(z_0) \neq 0$. In other words, if the principal part of the Laurent series of $f(z)$ about z_0 contains up to m-th power of $z - z_0$.

A pole of order one is called Simple pole.

Example

- For the function $f(z) = \dfrac{1}{(z-5)^2(z-1)^3}$, $z = 5$ is a pole of order '2' and $z = 1$ is a pole of order '3'.
- The function $\tan \dfrac{1}{z}$ has poles at $z = \dfrac{2}{n\pi}$ for $n = \pm 1, \pm 3, \ldots$

Classification of Singularities

Definition

If the principle part of $f(z)$ contains an infinite number of terms then the singularity $z = z_0$ is called an **essential singularity.**

Example

- For the function $e^{1/z}$ we can expand the function as

$$e^{1/z} = 1 + \frac{1}{z-0} + \frac{1}{(z-0)^2} + \dots$$

  Here $z = 0$ is an essential singularity of $e^{1/z}$. The limit point zero is an isolated essential singularity.

- For the function $f(z) = cosec(\frac{1}{z})$ poles are given by $z = 1/n\pi$. Limit point zero is an non-isolated singularity of $f(z)$.

Essential singularity

Theorem

Weierstrass: An analytic function comes arbitrarily close to any complex value in every neighborhood of an essential singularity.

Theorem

Picard: In each neighbourhood of an essential singularity a function assumes every finite value, except possibly one value, infinitely many times.

Singularities and Zeros

- We know that zeros of an analytic function $f(z)$ are isolated. Further, if z_0 is a zero of $f(z)$ then it is a pole of $1/f(z)$. Hence poles of an analytic function are also isolated.
- Hence if z_0 is a pole of a function $f(z)$ there is a neighbourhood of z_0 which contains neither a zero of $f(z)$ nor a singular point other than z_0 itself.

Meromorphic functions

Definition

A function which is analytic in the region Ω except at poles is called the **meromorphic** function.

Example

All rational functions $\tan z, \cot z, \tanh z \ldots$ etc are meromorphic functions.

If f is a meromorphic function on a bounded region R, then the poles of $\dfrac{f'}{f}$ in R all occur at points which are either zeros or poles of f.

Example

Let f be the function defined by $f(z) = \frac{(z+3)(z+1)}{z^2}$. Find the zero and poles of f, $\frac{1}{f}$, f' and $\frac{f'}{f}$.

Solution: Since $f(z) = \frac{(z+3)(z+1)}{z^2}$, we get

$$\frac{1}{f(z)} = \frac{z^2}{(z+3)(z+1)}, f'(z) = \frac{-(4z+6)}{z^3}, \frac{f'(z)}{f(z)} = \frac{-4(z+6)}{z(z+3)(z+1)}.$$

We therefore get the following table of zeros and poles, with the orders of the zeros and poles shown in brackets:

Meromorphic functions

	f	$1/f$	f'	f'/f
Zeros	$-3(1), -1(1)$	$0(2)$	$-3/2(1)$	$-3/2(1)$
Poles	$0(2)$	$-3(1), -1(1)$	$0(3)$	$0(1), -3(1), -1(1)$