One of the earliest work in graph theory is attributed to Euler who solved the famous puzzle called the Königsberg bridge problem. The interested reader is advised to see any text book on graph theory. Another important work in graph theory comes from the work of Kirchhoff who used them in the study of electric networks and proved the famous "matrix tree theorem".

4.1 Basic Graph Theory

Definition 4.1.1. *A graph (digraph) X consists of two sets V and E, where V, is a non-empty set, called the vertex set and E is called the edge set, and E is a multi-set consisting of unordered (ordered) pairs of elements of V. The graph X is also denoted as $X = (V, E)$.*

It is very convenient to have a pictorial representation of a graph. To do this, one uses points/nodes in place of vertices and lines in place of an edges. A graph whose vertices are labeled (not labeled) is called *a labeled (an unlabeled)* graph. Before proceeding further, we give examples of a few labeled and unlabeled graphs (see Figure 4.1).

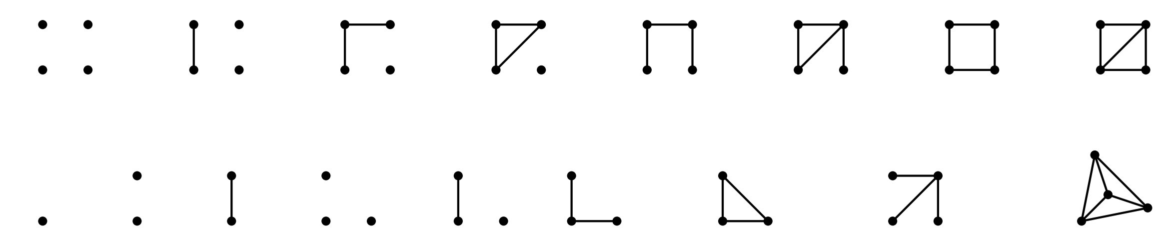
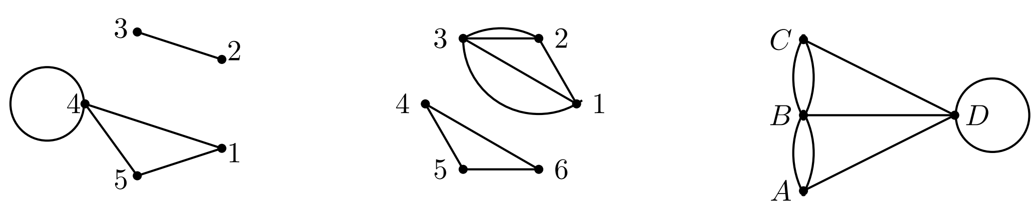
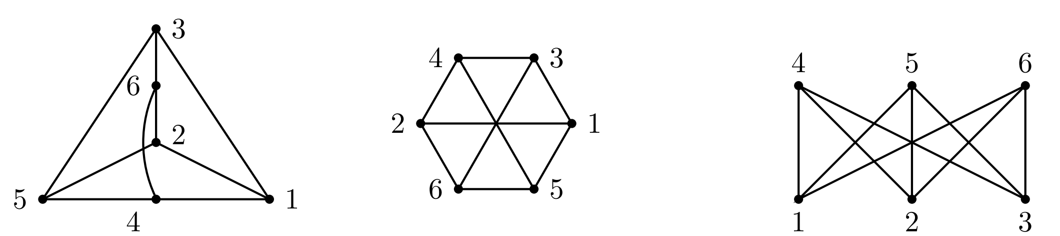


Figure 4.1: Examples of unlabeled graphs on 4 or less vertices



Examples of graphs that have loops or multiple edges



Examples of simple isomorphic graphs

Figure 4.2: Examples of labeled graphs

A graph $X = (V, E)$ is said to be a *directed graph* if there exist $u, v \in V$ such that $(u, v) \in E$ but $(v, u) \notin E$. We will mostly be concerned with graphs that are not directed. Therefore, we assume that whenever $(u, v) \in E$, the tuple $(v, u) \in E$ as well. We will use the notation $e = \{u, v\} \in E$ to say that the edge e is *incident* with u and v or the vertex u is *adjacent* to the vertex v, or vice-versa and in short, write it as $u \sim v$. An edge $e = \{u, u\}$ or (u, u) is called a *loop*. If two edges e_1 and e_2 are incident with a common vertex then the two edges are said to be *adjacent*. A graph is said to have *multiple* edges if the same edge appears more than once in the multiset E. A graph is called *simple* if it has no loops or multiple edges.

In these notes, unless stated otherwise, all our graphs will be labeled simple graphs having finite number of vertices (and hence finite number of edges).

2 is adjacent to 3 $\qquad$ e is adjacent to f

Figure 4.3: Examples of adjacency

Definition 4.1.2. *Let $X = (V, E)$ and $Y = (V', E')$ be two graphs. Then X is said to be isomorphic to Y if there exists a one-one onto function $f : V \longrightarrow V'$ such that $\{f(u), f(v)\} \in E'$ if, and only if, $\{u, v\} \in E$, for each $u, v \in V$.*

See Figure 4.2 for examples of isomorphic graphs. If two graphs, say X and Y, are isomorphic, then in graph theory, in place of writing $X \equiv Y$, we generally write $X = Y$. The numbers related with graphs that do not change under isomorphic transformations are called *graph invariant(s)*. Thus, note that if two graphs are isomorphic then they have the same number of vertices and the same number of edges. That is, the number of vertices and the number of edges are graph invariants. In general, a complete set of graph invariants are not known.

The *degree* of a vertex $v \in V$, denoted $\deg(v)$, is the number of edges incident at v. A vertex $v \in V$ is called an *isolated* vertex if $\deg(v) = 0$. A vertex $v \in V$ is called a *pendant* vertex if $\deg(v) = 1$. A graph is called a *null graph* if E is an empty set.

Observe that every edge is incident with two vertices and therefore, contributes 2 to the sum of degrees. That is, if $e = \{u, v\} \in E$ then e is counted once in $\deg(u)$ and is counted again in $\deg(v)$. Hence, one obtains the following result, famously called the handshake lemma.

Lemma 4.1.3. *Let $X = (V, E)$ be a graph. Then* $\sum\limits_{v \in V} \deg(v) = 2|E|$.

As corollaries to Lemma 4.1.3, one immediately obtains a few results. To state these results, we will need the following definition.

Definition 4.1.4. *Let $X = (V, E)$ be a graph. Then X is said to be a cubic graph if* $\deg(v) = 3$, *for all $v \in V$. A vertex $v \in V$ is called a vertex of* odd (even) degree *if* $\deg(v)$ *is odd (even).*

Corollary 4.1.5. *Let $X = (V, E)$ be a graph. Then the number of vertices with odd degree is even.*

Corollary 4.1.6. *Let $X = (V, E)$ be a cubic graph. Then the number of vertices is even.*

Definition 4.1.7. *Let $X = (V, E)$ be a graph.*

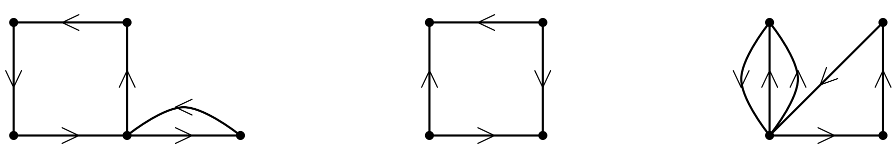
1. *A walk in X is a sequence of vertices, say $[v_0, v_1, \ldots, v_k]$, such that v_{i-1} is adjacent to v_i, for $i = 1, 2, \ldots, k$.*
2. *The walk $[v_0, v_1, \ldots, v_k]$ is called closed, if $v_0 = v_k$ and open, otherwise.*
3. *The walk $[v_0, v_1, \ldots, v_k]$ is called a trail if all the edges are distinct.*
4. *The walk $[v_0, v_1, \ldots, v_k]$ is called a path if all the vertices, and hence the edges, are distinct.*
5. *A cycle is a path in which the end vertices are allowed to be the same.*
6. *The length of the walk $[v_0, v_1, \ldots, v_k]$ is defined to be equal to k.*

We also define a distance between two vertices of a graph as follows.

Definition 4.1.8. *Let $X = (V, E)$ be a graph. Then X is said to be a*

1. *connected graph if, for each $u, v \in V$, there exists a path from u to v.*
2. *disconnected, if it is not connected. That is, there exist vertices $u, v \in V$ such that there is no path from u to v.*

In case of directed graph, one talks of strongly connected in place of connected. That is, a directed graph $X = (V, E)$ is said to be strongly connected if for each $u, v \in V$, there is a path from u to v. Note that in Figure 4.4, the directed graph X_1 and X_3 are strongly connected but X_2 is not a strongly connected directed graph.



X_1 X_2 X_3

Figure 4.4: Examples of directed graphs

We also have the following definition that gives a notion of metric.

Definition 4.1.9. *Let $X = (V, E)$ be a graph and let $u, v \in V$. Then, the distance between u and v, denoted $d(u, v)$, in X is the length of the shortest path between u and v, if there exists a path between u and v, else $d(u, v) = \infty$.*

A shortest path from u to v is called a geodesic and the diameter of a graph X is the length of the largest geodesic.

With the definitions as above, one has the following result in graph theory.

Proposition 4.1.10. *Let $X = (V, E)$ be a non-null graph. Then X is disconnected if and only if the vertex set V can be partitioned into two parts, say V_1, V_2, such that if $e = \{u, v\} \in E$ then either both $u, v \in V_1$ or both $u, v \in V_2$.*

Proof. Let us assume that there exists a partition of V into two parts, say V_1 and V_2, satisfying the properties stated in the proposition. Since V_1 and V_2 are non-empty, there exists $u \in V_1$ and $v \in V_2$. We claim that there is no path in X joining the vertices u and v. For if, there is a path from u to v, then there will be at least one edge, say $e = \{x, y\} \in E$ such that $x \in V_1$ and $y \in V_2$. This contradicts the assumption that either both $x, y \in V_1$ or both $x, y \in V_2$.

Conversely, let us assume that X is a disconnected graph. Now, fix a vertex, say $u \in V$ and define $V_1 = \{x \in V : d(u, x) < \infty\}$. Since X is disconnected $V_1 \subsetneq V$ and hence the set $V_2 = V \setminus V_1$ is a non-empty subset of V. Clearly the two sets V_1 and V_2 give a partition of V and there is no edge joining a vertex in V_1 with a vertex in V_2. Hence, this completes the proof of the proposition. ■

We end this section, with a small list of well known graphs and a set of exercises.

Definition 4.1.11.

1. *Let $X = (V, E)$ be a graph on $|V| = n$ vertices, with $V = \{v_1, v_2, \ldots, v_n\}$. Then X is said to be a*
   - (a) *null graph, denoted $\mathbf{0}_n$, if $E = \emptyset$.*
   - (b) *path graph, denoted P_n, if $E = \{ \{v_i, v_{i+1}\} : 1 \le i \le n-1\}$.*
   - (c) *cycle graph, denoted C_n, if $E = \{ \{v_i, v_{i+1}\} : 1 \le i \le n-1\} \cup \{\{v_1, v_n\}\}$.*
   - (d) *tree if X is connected and has no cycle.*
   - (e) *complete graph, denoted K_n, if $E = \{ \{v_i, v_j\} : 1 \le i < j \le n\}$.*
   - (f) *d-regular graph if $\deg(v_i) = d$ for all $i = 1, 2, \ldots, n$. Note that a cycle graph is a 2-regular graph, whereas the complete graph K_n is a $(n-1)$-regular graph.*
2. *A graph $X = (V, E)$ is said to be a bipartite graph, if V is a disjoint union of two sets, say V_1 and V_2 such that every edge of X is incident with a vertex in V_1 and the other with a vertex in V_2.*
3. *A graph $X = (V, E)$ is said to be a complete bipartite graph, if $V = \{v_1, \ldots, v_m, u_1, \ldots, u_n\}$ and $E = \{\{v_i, u_j\} : 1 \le i \le m,\ 1 \le j \le n\}$. This graph is denoted by $K_{m,n}$.*

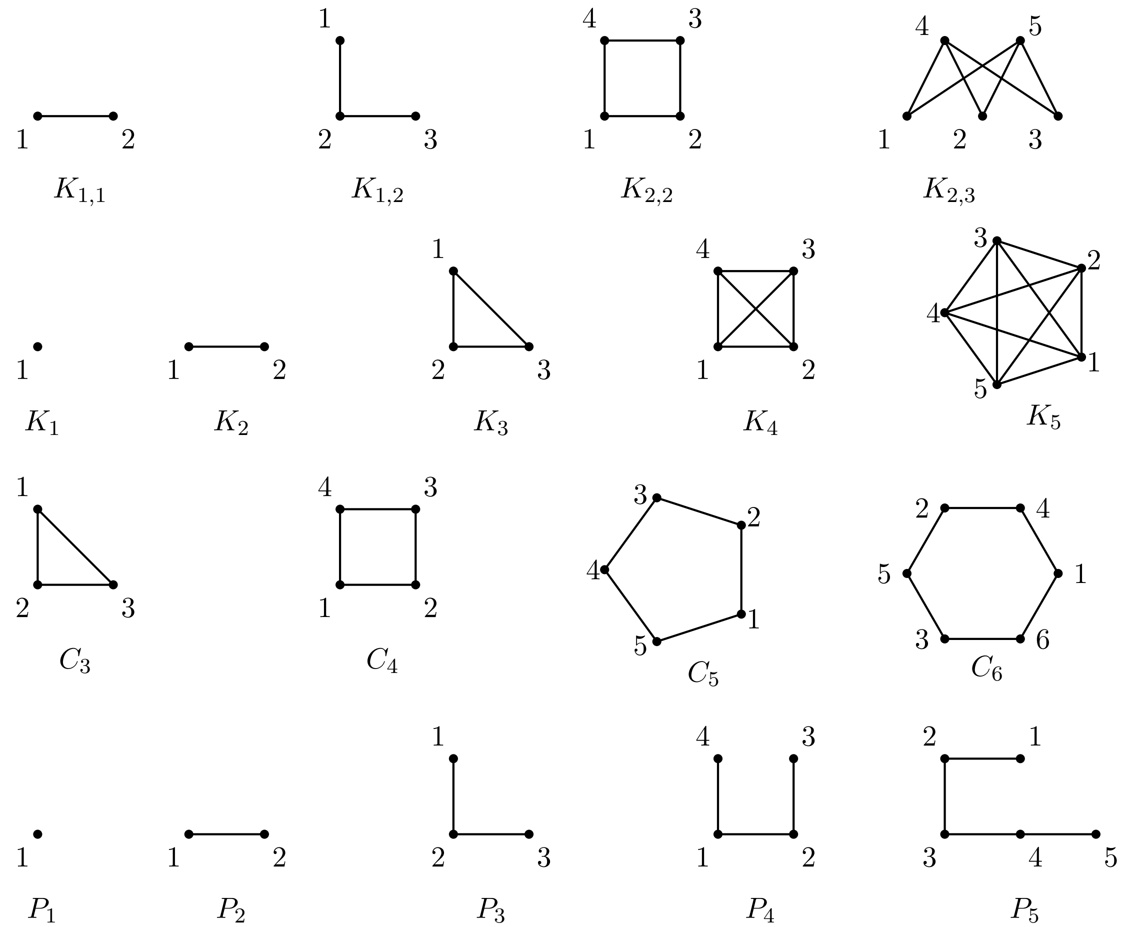


Figure 4.5: Some well known family of graphs

4.2 Graph Operations

A graph $Y = (V', E')$ is said to be a *subgraph* of a graph $X = (V, E)$ if $V' \subset V$ and $E' \subset E$. One also states this by saying that the graph X is a *supergraph* of the graph Y. A subgraph $Y = (V', E')$ of $X = (V, E)$ is said to be a *spanning subgraph* if $V' = V$ and is called an *induced subgraph* if, for each $u, v \in V' \subset V$, the edge $\{u, v\} \in E'$ whenever $\{u, v\} \in E$. In this case, the set V' is said to induce the subgraph Y and this is denoted by writing $Y = X[V']$. With the definitions as above, one observes the following:

1. Every graph is its own subgraph.

2. If Z is a subgraph of Y and Y is a subgraph of X then Z is also a subgraph of X.

3. A single vertex of a graph is also its subgraph.

4. A single edge of a graph together with its incident vertices is also its subgraph.

We also have the following graph operations.

Definition 4.2.1. *1. Let $X_1 = (V_1, E_1)$ and $X_2 = (V_2, E_2)$ be two graphs. Then the*