Lecture 2 Essential Ordinary Differential Equations

In this lecture, we recall some methods of solving first-order IVP in ODE (separable and linear) and homogeneous second-order linear ODEs with constant coefficients. These results will be useful while solving linear homogeneous PDEs using the variables separable method in the subsequent modules (cf., Module 5, Module 6 and Module 7). Some fundamental results on existence, uniqueness and continuous dependence of solutions on given data will also be discussed.

First-Order ODEs: A first-order ODE is separable if it can be written in the form

$$f(y)\frac{dy}{dx} = g(x), \tag{1}$$

where y is an unknown function of the independent variable x.

Integrate both sides (if possible) with respect to x to have

$$\begin{aligned} \int f(y)\frac{dy}{dx}\,dx &= \int g(x)\,dx \\ \Longrightarrow \quad \int f(y)\,dy &= \int g(x)\,dx, \end{aligned}$$

and from which we find solutions $y(x)$.

Consider a first order linear **nonhomogeneous equation** ODE in the standard form

$$y'(x) + p(x)y(x) = q(x). \tag{2}$$

When $q(x) = 0$, the resulting equation

$$y'(x) + p(x)y(x) = 0$$

is called **homogeneous equation** which can be put in a separable form

$$\frac{dy}{y} = -p(x)\,dx, \qquad (y \neq 0).$$

Its solution is thus given by

$$y_h(x) = C\,\exp\left(-\int p(x)\,dx\right),$$

where C is an arbitrary constant. To solve (2), an integrating factor is given by

$$\boxed{\mu(x) = \exp\left(\int p(x)\,dx\right).}$$

The general solution of (2) is given by

$$\boxed{y(x) = \frac{1}{\mu(x)}\left\{\int \mu(x)q(x)dx + C_1\right\}.} \tag{3}$$

Example 1. *Solve* $(1+x^2)y' + 2xy = 5x^4$.

Solution. *Putting the equation into the standard form (2), we have*

$$y' + \{2x/(1+x^2)\}y = 5x^4/(1+x^2). \tag{4}$$

Here, $p(x) = 2x/(1+x^2)$ *and* $q(x) = \frac{5x^4}{(1+x^2)}$. *An integrating factor for (2) is*

$$\mu(x) = \exp\left[\int \{2x/(1+x^2)\}dx\right] = \exp[\log(1+x^2)] = 1+x^2.$$

Multiplying both side of (4) by $\mu(x)$, we obtain

$$\frac{d}{dx}[\mu(x)y(x)] = \mu(x)q(x) = 5x^4.$$

Integrate both side of the above equation to have

$$\begin{aligned} & \mu(x)y(x) = x^5 + C \\ \implies \quad & y(x) = (x^5 + C)/(1+x^2). \end{aligned}$$

Second-Order Linear ODEs with Constant Coefficients: We recall some basic results of the homogeneous second-order linear ODE of the form

$$ay''(x) + by'(x) + cy(x) = 0, \tag{5}$$

where the coefficients a, b, and c are real constants with $a \neq 0$. Let m_1 and m_2 be the roots of the associated auxiliary equation

$$am^2 + bm + c = 0.$$

- If m_1 and m_2 are real and distinct ($b^2 - 4ac > 0$), then the general solution of (5) is

$$\boxed{y(x) = c_1e^{m_1x} + c_2e^{m_2x}.}$$

- If $m_1 = m_2 = m$ ($b^2 - 4ac = 0$), then the general solution of (5) is

$$\boxed{y(x) = c_3e^{mx} + c_4xe^{mx}.}$$

- If $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$ ($b^2 - 4ac < 0$), then the general solution of (5) is

$$\boxed{y(x) = e^{\alpha x}[c_5 \cos(\beta x) + c_6 \sin(\beta x)].}$$

Here, c_i, $i = 1, 2, 3, 4, 5, 6$ are arbitrary constants.

On Existence and Uniqueness of IVP: Consider the following IVPs:

$$|y'| + 2|y| = 0, \quad y(0) = 1. \tag{6}$$

$$y'(x) = x, \quad y(0) = 1. \tag{7}$$

$$xy' = y - 1, \quad y(0) = 1. \tag{8}$$

Note that the IVP (6) has no solution, the problem (7) has precisely one solution, namely $y = \frac{1}{2}x^2 + 1$ and the problem (8) has infinitely many solutions, namely $y = 1 + cx$, where c is an arbitrary constant. From the above three IVPs, we observe that an IVP

$$y'(x) = f(x, y), \quad y(x_0) = y_0$$

may have none, precisely one, or more than one solution. This leads to the following fundamental results.

Theorem 2. (Existence) *Let $R : |x - x_0| < a,\ |y - y_0| < b$ be a rectangle. If $f(x, y)$ is continuous and bounded in R i.e., there is a number K such that*

$$|f(x, y)| \leq K \quad \forall (x, y) \in R,$$

then the IVP

$$y'(x) = f(x, y), \quad y(x_0) = y_0 \tag{9}$$

has at least one solution $y(x)$. This solution is defined for all x in the interval

$$|x - x_0| < \alpha, \quad \text{where} \quad \alpha = \min\{a, \frac{b}{K}\}.$$

Theorem 3. (Uniqueness) *Let $R : |x - x_0| < a,\ |y - y_0| < b$ be a rectangle. If $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous and bounded in R i.e., there exist two number K and M such that*

$$|f(x, y)| \leq K \quad \forall (x, y) \in R, \tag{10}$$

$$\left|\frac{\partial f}{\partial y}\right| \leq M \quad \forall (x, y) \in R, \tag{11}$$

then the IVP (9) has a unique solution $y(x)$. This solution is defined for all x in the interval

$$|x - x_0| < \alpha, \quad \text{where} \quad \alpha = \min\{a, \frac{b}{K}\}.$$

Example 4. *Let $R : |x| < 5, \ |y| < 3$ be the rectangle. Consider the IVP*

$$y' = 1 + y^2, \quad y(0) = 0$$

over R.

Here, $a = 5, \ b = 3$. Then

$$\max_{(x,y)\in R} |f(x,y)| = \max_{(x,y)\in R} |1 + y^2| \le 10 (= K),$$

$$\max_{(x,y)\in R} \left|\frac{\partial f}{\partial y}\right| = \max_{(x,y)\in R} 2|y| \le 6 (= M).$$

$$\alpha = \min\{a, \frac{b}{K}\} = \min\{5, \ \frac{3}{10}\} = 0.3 < 5.$$

Note that the solution of the IVP is $y = \tan x$. This solution is valid in the interval $|x| < 0.3$ in stead of the entire interval $|x| < 5$. It is easy check that the solution $y = tanx$ is discontinuous at $x = \pm\pi/2$, and hence, there is no continuous solution valid in the entire interval $|x| < 5$.

The conditions in Theorem 3 are sufficient conditions rather than necessary ones, and can be lessened. By the mean value theorem of differential calculus, we have

$$f(x, y_2) - f(x, y_1) = (y_2 - y_1)\frac{\partial f}{\partial y}(x, \eta),$$

where $(x, y_1), \ (x, y_2) \in R$ and η lies between y_1 and y_2. In view of the condition

$$\left|\frac{\partial f}{\partial y}\right| \le M \quad \forall (x, y) \in R,$$

it follows that

$$|f(x, y_2) - f(x, y_1)| \le M|y_2 - y_1|, \tag{12}$$

which is known as a *Lipschitz condition.* Thus, the condition (11) can be weakened to obtain the following existence and uniqueness result.

Theorem 5. (Picard's Theorem) *Let $R : |x - x_0| < a, \ \ |y - y_0| < b$ be a rectangle. Let $f(x, y)$ be continuous and bounded in R i.e., there exists a number K such that*

$$|f(x, y)| \le K \ \ \forall (x, y) \in R.$$

Further, let f satisfy the Lipschitz condition *with respect to y in R, i.e., there exists a number M such that*

$$|f(x, y_2) - f(x, y_1)| \le M|y_2 - y_1| \quad \forall (x, y_1), \ (x, y_2) \in R. \tag{13}$$

Then, the IVP (9) has a unique solution $y(x)$. This solution is defined for all x in the interval

$$|x - x_0| < \alpha, \quad \textit{where} \quad \alpha = \min\{a, \frac{b}{K}\}.$$

Note that the continuity of f is not enough to guarantee the uniqueness of the solution which can be seen from the following example.

Example 6. (Nonuniqueness) *Consider the IVP:*

$$y' = \sqrt{|y|}, \quad y(0) = 0.$$

Note that $f(x, y) = \sqrt{|y|}$ is continuous for all y. However,

$$y \equiv 0 \quad \text{and} \quad \hat{y} = \begin{cases} x^2/4, & x \geq 0 \\ -x^2/4, & x < 0. \end{cases}$$

are two solutions of the given IVP. The uniqueness fails because the *Lipschitz condition* (13) is violated in any region which include the line $y = 0$. With $y_1 = 0$ and $y_2 > 0$, we note that

$$\frac{|f(x, y_2) - f(x, y_1)|}{|y_2 - y_1|} = \frac{|f(x, y_2) - f(x, 0)|}{|y_2|} = \frac{\sqrt{y_2}}{y_2} = \frac{1}{\sqrt{y_2}},$$

which can be made large by choosing $y_2 \to 0$. Thus, it is not possible to find a fixed constant M such that the condition (13) holds, and hence the IVP has a solution but it is not unique.

Next, we generalize the above result to a system of n first order ordinary differential equations in n unknowns of the form

$$\frac{dy_i(x)}{dx} = f_i(x, y_1, \ldots, y_n), \quad i = 1, \ldots, n, \tag{14}$$

satisfying the initial conditions

$$y_1(x_0) = y_1^0, \ldots, y_n(x_0) = y_n^0, \tag{15}$$

where $y_1^0, \ldots, y_n^0$ are the given initial values.

The fundamental result concerning the existence and uniqueness of solution of the system (14)-(15) is essentially the same as Theorem 5.

Theorem 7. *Let Q be a box in $\mathbb{R}^{n+1}$ defined by*

$$Q : |x - x_0| < a, \ \ |y_1 - y_1^0| < b_1, \ldots, |y_n - y_n^0| < b_n.$$

Let each of the functions $f_1, \ldots, f_n$ be continuous and bounded in Q, and satisfy the following Lipschitz condition with respect to the variables $y_1, y_2, \ldots, y_n$, i.e., there exists constants $L_1, \ldots, L_n$ such that

$$|f(x, y_1^1, \ldots, y_n^1) - f(x, y_1^2, \ldots, y_n^2)| \leq L_1|y_1^1 - y_1^2| + \cdots + L_n|y_n^1 - y_n^2|$$

for all pairs of points $(x, y_1^1, \ldots, y_n^1), (x, y_1^2, \ldots, y_n^2) \in Q$. Then there exists a unique set of functions $y_1(x), \ldots, y_n(x)$ defined for x in some interval $|x - x_0| < h, \quad 0 < h < a$ such that $y_1(x), \ldots, y_n(x)$ solve (14)-(15).

Practice Problems

1. Determine whether the given differential equation is separable.
   $(a)\ \frac{dy}{dx} = \frac{ye^{x+y}}{x^2+y}$; $(b)\ x\frac{dy}{dx} = 1 + y^2$; $(c)\ \frac{dy}{dx} = \sin(x + y)$.

2. Solve the following first-order linear equations subject to the given conditions:
   $(a)\ \frac{dy}{dx} + \frac{y}{x} = 1, \quad y(2) = 1$; $(b)\ 4\frac{dy}{dx} + 3xy = 5, \quad y(0) = 1$; $(c)\ \sin x\frac{dy}{dx} + y\cos x = x\sin x, \quad y(\pi/2) = 2$.

3. Find the general solution of the following second-order homogeneous linear ODEs.
   $(a)\ \frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0$; $(b)\ \frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$; $(c)\ \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = 0$.

4. Does $f(x, y) = |x| + |y|$ satisfy a Lipschitz condition in the xy-plane? Does $\partial f/\partial y$ exist?

5. Find all solutions of the IVP $\frac{dy}{dx} = 2\sqrt{y}, \ y(1) = 0$.