Chapter 7
Characteristic functions

Lectures 31 - 33
In this chapter, we introduce the notion of characteristic function of a random variable and study its properties. Characteristic function serves as an important tool for analyzing random phenomenon.

Definition 7.1 (Characteristic functions) The characteristic function of a random variable $X$ is defined as

$$\Phi_X(t) = E e^{itX}, \quad t \in \mathbb{R}. $$

(where $Ee^{itX} = E \cos tX + iE \sin tX$)

Example 7.0.43 Let $X \sim \text{Bernoulli}(p)$. Then

$$\phi_X(t) = (1 - p) + pe^{it}.$$ 

Example 7.0.44 Let $X \sim \text{exponential}(\lambda)$. Then

$$\Phi_X(t) = E e^{itX} = \lambda \int_0^\infty e^{itx} e^{-\lambda x} \, dx$$

$$= \lambda \left( \int_0^\infty \cos txe^{-\lambda x} \, dx \right)$$

$$= i \lambda \left( \int_0^\infty \sin txe^{-\lambda x} \, dx \right)$$

$$= \lambda \left( \frac{\lambda^2 + t^2}{\lambda^2 + t^2} \right)$$

$$= \frac{\lambda (\lambda + it)}{\lambda^2 + t^2}, \quad t \in \mathbb{R}.$$ 

Theorem 7.0.34 For any random variable $X$, its characteristic function $\phi_X(\cdot)$ is uniformly continuous on $\mathbb{R}$ and satisfies

(i) $\Phi_X(0) = 1$

(ii) $|\Phi_X(t)| \leq 1$

(iii) $\Phi_X(-t) = \overline{\Phi_X(t)}$, where for $z$ a complex number, $\overline{z}$ denote the conjugate.

Proof: We prove (iii), (i) and (ii) are exercises.

$$\Phi_X(-t) = E e^{-itX} = E \cos tX - iE \sin tX$$

$$= \overline{E \cos tX + iE \sin tX}$$

$$= \overline{\Phi_X(t)}.$$

Now we show that $\Phi_X$ is uniformly continuous. Consider
\[ |\Phi_X(t+h) - \Phi_X(t)| = |E(e^{it(t+h)}X - e^{itX})|, \]
\[ \leq E|e^{ithX} - 1| \]
\[ = E\sqrt{2(1 - \cos(hX))} \]
\[ = 2E|\sin\left(\frac{hX}{2}\right)| \]

Using Dominated Convergence theorem, \( \Phi_X(t+h) \rightarrow \Phi_X(t) \) uniformly in \( t \) as \( h \rightarrow 0 \). This imply that \( \Phi_X \) is uniformly continuous.

**Theorem 7.0.35** If the random variable \( X \) has finite moments upto order \( n \). Then \( \Phi \) has continuous derivatives upto order \( n \). More over
\[ i^k E X^k = \Phi_X^{(k)}(0), \quad k = 1, 2, \ldots, n. \]

**Proof.** Consider
\[ \Phi_X(t+h) - \Phi_X(t) = E\left[e^{itX}(e^{ihX} - 1)\right] \]

since \( |e^{ithX} - 1| \leq |hx| \), we get
\[ E\left[|e^{itX}(e^{ihX} - 1)|\right] \leq E|X| < \infty. \]

Hence by Dominated Convergence theorem
\[ \lim_{h \rightarrow 0} E\left[\frac{e^{itX}(e^{ihX} - 1)}{h}\right] = E[iXe^{itX}]. \]

Therefore
\[ \Phi_X'(t) = E[iXe^{itX}]. \]

Put \( t = 0 \), we get
\[ \Phi_X^{(1)}(0) = iEX. \]

For higher order derivatives, repeat the above arguments.

**Theorem 7.0.36** (Inversion theorem) Let \( X \) be a random variable with distribution function \( F \) and characteristic function \( \phi_X(\cdot) \). Then
\[ F(b) - F(a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ita} - e^{-itb}}{it} \Phi_X(t) dt, \]

whenever \( a, b \) are points of continuity of \( F \).

**Proof.** Consider
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ita} - e^{-itb}}{it} \Phi_X(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ita} - e^{-itb}}{it} E e^{itX} dt \\
= \frac{1}{2\pi} E \int_{-\infty}^{\infty} \frac{e^{-ita} - e^{-itb}}{it} e^{itX} dt = E \int_{-\infty}^{\infty} \frac{e^{it(X-a)} - e^{it(X-b)}}{2\pi it} dt. \tag{7.0.1}
\]

The second equality follows from the change of order of integration. Now

\[
\int_{-\infty}^{0} \frac{e^{it(X-a)} - e^{it(X-b)}}{2\pi it} dt = \int_{0}^{\infty} \frac{e^{-it(X-a)} - e^{-it(X-b)}}{2\pi it} dt \tag{7.0.2}
\]

Hence, using \(2i\sin \theta = e^{i\theta} - e^{-i\theta}\), we have

\[
\int_{-\infty}^{\infty} \frac{e^{it(X-a)} - e^{it(X-b)}}{2\pi it} dt = \frac{1}{\pi} \int_{0}^{\infty} \frac{\sin t(X-a)}{t} dt - \frac{1}{\pi} \int_{0}^{\infty} \frac{\sin t(X-b)}{t} dt \tag{7.0.3}
\]

Using

\[
\int_{0}^{\infty} \frac{\sin ax}{x} dx = \frac{\pi}{2} \text{sgn}(a)
\]

we get

\[
\frac{1}{\pi} \int_{0}^{\infty} \frac{\sin t(X-a)}{t} dt = \begin{cases} 
\frac{1}{2} & \text{if } X > a \\
0 & \text{if } X = a \\
-\frac{1}{2} & \text{if } X < a,
\end{cases} \tag{7.0.4}
\]

where

\[
\text{sgn}(\alpha) = \begin{cases} 
-1 & \text{if } \alpha < 0 \\
0 & \text{if } \alpha = 0 \\
1 & \text{if } \alpha > 0.
\end{cases}
\]

Similarly, the other integral. Combining (7.0.1), (7.0.3) and (7.0.4), we complete the proof.

**Theorem 7.0.37** (Uniqueness Theorem)

Let \(X_1, X_2\) be two random variables such that \(\Phi_{X_1} \equiv \Phi_{X_2}\). Then \(X_1, X_2\) have the same distribution.

**Proof:** Using Inversion theorem, we have

\[
F_1(b) - F_1(a) = F_2(b) - F_2(a)
\]

for all \(a, b \in \mathbb{R}\) such that \(F_1, F_2\) are continuous at \(a\) and \(b\).

Now let \(a \to -\infty\), we have

\[
F_1(b) = F_2(b)
\]

for all \(b\) at which \(F_1\) and \(F_2\) are continuous.

Therefore

\[
F_1 \equiv F_2 \text{ (Exercise)}
\]
Now we state the following theorem whose proof is beyond the scope of this course.

**Theorem 7.0.38 (Continuity Theorem)** Let \(X_n, X\) be random variables on \((\Omega, \mathcal{F}, P)\) such that,

\[
\lim_{n \to \infty} \Phi_{X_n}(t) = \Phi_X(t), \quad t \in \mathbb{R}.
\]

Then \(F_{X_n}(x) \rightarrow F_X(x)\) for all \(x \in \mathbb{R}\) such that \(F\) is continuous at \(x\).

**Proof:**
A detailed proof is beyond the scope of this course but I will give an idea of the proof. Choose a standard normal random variable \(Y\) which is independent of \(X\) and \(X_n, n \geq 1\). For \(x\) such that \(P\{X = x\}, \quad \epsilon > 0\) and \(y < x\) such that \(P\{X_n = y\} = P\{X = y\} = 0, n \geq 1\), using inversion theorem we have

\[
P\{y \leq X_n + \epsilon, Y \leq x\} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-it\epsilon} - e^{-it\epsilon}}{it} \Phi_{X_n}(t)e^{-t^2/2} dt
\]
and

\[
P\{y \leq X + \epsilon, Y \leq x\} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-it\epsilon} - e^{-it\epsilon}}{it} \Phi_X(t)e^{-t^2/2} dt
\]

Now by letting first \(n \to \infty\) and \(y \to \infty\) then \(\epsilon \to 0\) and finally, we get

\[
\lim_{n \to \infty} P\{X_n \leq x\} = P\{X \leq x\}
\]

Here note that above mentioned limits need justification.