A topological group is a topological space which is also a group such that the group operations (multiplication and inversion) are continuous. They arise naturally as continuous groups of symmetries of topological spaces. A case in point is the group $SO(3, \mathbb{R})$ of rotations of $\mathbb{R}^3$ about the origin which is a group of symmetries of the sphere $S^2$. Many familiar examples of topological spaces are in fact topological groups. The most basic example of course is the real line with the group structure given by addition. Other obvious examples are $\mathbb{R}^n$ under addition, the multiplicative group of unit complex numbers $S^1$ and the multiplicative group $\mathbb{C}^*$. In the previous lectures we have seen that the group $SO(n, \mathbb{R})$ of orthogonal matrices with determinant one and the group $U(n)$ of unitary matrices are compact. In this lecture we initiate a systematic study of topological groups and take a closer look at some of the matrix groups such as $SO(n, \mathbb{R})$ and the unitary groups $U(n)$.

**Definition 5.1:** A topological group is a group which is also a topological space such that the singleton set containing the identity element is closed and the group operation

$$G \times G \rightarrow G$$

$$(g_1, g_2) \mapsto g_1g_2$$

and the inversion $j : G \rightarrow G$ given by $j(g) = g^{-1}$ are continuous, where $G \times G$ is given the product topology.

We leave it to the reader to prove that a topological group is a Hausdorff space. It is immediate that the following maps of a topological group $G$ are continuous:

1. Given $h \in G$ the maps $L_h : G \rightarrow G$ and $R_h : G \rightarrow G$ given by $L_h(g) = hg$ and $R_h(g) = gh$. These are the left and right translations by $h$.

2. The inner-automorphism given by $g \mapsto hgh^{-1}$ which is a homeomorphism.

Note that the determinant map is a continuous group homomorphism from $GL_n(\mathbb{R}) \rightarrow \mathbb{R} - \{0\}$. The image is surjective from which it follows that $GL_n(\mathbb{R})$ is disconnected as a topological space.

**Theorem 5.1:** The connected component of the identity in a topological group is a subgroup.

**Proof:** Let $G_0$ be the connected component of $G$ containing the identity and $h, k \in G_0$ be arbitrary. The set $h^{-1}G_0$ is connected and contains the identity and so $G_0 \cup h^{-1}G_0$ is also connected. Since $G_0$ is a component, we have $G_0 \cup h^{-1}G_0 = G_0$ which implies $h^{-1}G_0 \subset G_0$. In particular $h^{-1}k$ belongs to $G_0$ from which we conclude that $G_0$ is a subgroup.

Interesting properties of topological groups arise in connection with quotients:

**Theorem 5.2:** Suppose that $G$ is a topological group and $K$ is a subgroup and the coset space $G/K$ is given the quotient topology then

1. If $K$ and $G/K$ are connected then $G$ is connected.

2. If $K$ and $G/K$ are compact then $G$ is compact.
Proof: If $G$ is connected then so is $G/K$ since the quotient map $\eta: G \to G/K$ is a continuous surjection. To prove the converse suppose that $K$ and $G/K$ are connected and $f: G \to \{0, 1\}$ be an arbitrary continuous map. We have to show that $f$ is constant. The restriction of $f$ to $K$ must be constant and since each coset $gK$ is connected, $f$ must be constant on $gK$ as well taking value $f(g)$. Thus we have a well defined map $\tilde{f}: G/K \to \{0, 1\}$ such that $\tilde{f} \circ \eta = f$. By the fundamental property of quotient spaces, it follows that $\tilde{f}$ is continuous and so must be constant since $G/K$ is connected. Hence $f$ is also constant and we conclude that $G$ is connected.

Since we shall not need (2), we shall omit the proof. A proof is available in [12], p. 109.

**Theorem 5.3:** The groups $SO(n, \mathbb{R})$ are connected when $n \geq 2$.

**Proof:** It is clear that $SO(2, \mathbb{R})$ is connected (why?). Turning to the case $n \geq 3$, we consider the action of $SO(n, \mathbb{R})$ on the standard unit sphere $S^{n-1}$ in $\mathbb{R}^n$ given by

$$(A, v) \mapsto Av,$$

where $A \in SO(n, \mathbb{R})$ and $v \in S^{n-1}$. It is an exercise for the student to check that this group action is transitive and that the stabilizer of the unit vector $\hat{e}_n$ is the subgroup $K$ consisting of all those matrices in $SO(n, \mathbb{R})$ whose last column is $\hat{e}_n$. The subgroup $K$ is homeomorphic to $SO(n-1, \mathbb{R})$ and so, by induction hypothesis, is connected. By exercise 3, the quotient space $SO(n, \mathbb{R})$ is homeomorphic to $S^{n-1}$ which is connected. So the theorem can be applied with $G = SO(n, \mathbb{R})$, $H = SO(n-1, \mathbb{R})$ and $G/H$ is the sphere $S^{n-1}$ with $n \geq 2$.

**Theorem 5.4:** If $G$ is a connected topological group and $H$ is a subgroup which contains a neighborhood of the identity then $H = G$. In particular, an open subgroup of $G$ equals $G$.

**Proof:** Let $U$ be the open neighborhood of the identity that is contained in $H$ and $h \in H$ be arbitrary. Since multiplication by $h$ is a homeomorphism, the set $Uh = \{uh/u \in U\}$ is also open and also contained in $H$. Hence the set

$$L = \bigcup_{h \in H} Uh$$

is open and contained in $H$. Since $U$ contains the identity element, $H \subseteq L$ and we conclude that $H$ is open. Our job will be over if we can show that $H$ is closed as well. Let $x \in \overline{H}$ be arbitrary. Since the neighborhood $Ux$ of $x$ contains a point $y \in H$, there exists $u \in U$ such that $y = ux$ which, in view of the fact that $U \subseteq H$, implies $x \in H$. Hence $\overline{H} = H$.

**Theorem 5.5:** Suppose $G$ is a connected topological group and $H$ is a discrete normal subgroup of $G$ then $H$ is contained in the center of $G$.

**Proof:** Since $H$ is discrete, the identity element is not a limit point of $H$ and so there is a neighborhood $U$ of the identity such that $U \cap H = \{1\}$. We may assume $U$ has the property that if $u_1, u_2$ are in $U$ then the product $u_1^{-1}u_2$ is in $U$. This follows from the continuity of the group operation and a detailed verification is left as an exercise. It is easy to see that if $h_1$ and $h_2$ are two distinct elements of $H$ then

$$U_{h_1} \cap U_{h_2} = \emptyset.$$
Fix $h \in H$ and consider now the set $K$ given by

$$K = \{g \in G \mid gh = hg\}$$

We shall show that the subgroup $K$ contains a neighborhood of the identity. Pick a neighborhood $V$ of the identity such that $V = V^{-1}$ and $(hVh^{-1}V) \cap H = \{1\}$. Then for any $g \in V$, we have on the one hand

$$hgh^{-1}g^{-1} \in hVh^{-1}V$$

and on the other hand $hgh^{-1}g^{-1} \in H$ since $H$ is normal. Hence $hgh^{-1}g^{-1} \in (hVh^{-1}V) \cap H = \{1\}$ which shows that $g$ belongs to $K$ and $K$ contains a neighborhood of the unit element. We may now invoke the previous theorem. \hfill \Box

**Remark:** The result is false if the hypothesis of normality of $H$ is dropped. For example consider a cube in $\mathbb{R}^3$ with center at the origin and $H$ be the subgroup of $G = SO(3, \mathbb{R})$ that map the cube to itself. Then $H$ is the symmetric group on four letters (proof?). Clearly $H$ is not in the center of $G$.

**Exercises**

1. Show that in a topological group, the connected component of the identity is a normal subgroup.

2. Show that the action of the group $SO(n, \mathbb{R})$ on the sphere $S^{n-1}$ given by matrix multiplication is transitive. You need to employ the Gram-Schmidt theorem to complete a given unit vector to an orthonormal basis.

3. Suppose a group $G$ acts transitively on a set $S$ and $x, y$ are a pair of points in $S$ and $y = gx$. Then the subgroups stab $x$ and stab $y$ are conjugates and $g^{-1}(\text{stab } y)g = \text{stab } x$.

   (i) Show that the map $\overline{\phi} : G/\text{stab } x \rightarrow S$ given by $\overline{\phi}(\overline{g}) = gx$ is well-defined, bijective and $\overline{\phi} \circ \eta = \phi$.

   (ii) Suppose that $S$ is a topological space, $G$ is a topological group and the action $G \times S \rightarrow S$ is continuous. Show that the map $\overline{\phi}$ is continuous.

   (iii) Deduce that if $G$ is compact and $S$ is Hausdorff then $G/\text{stab } x$ and $S$ are homeomorphic.

4. Examine whether the map $\phi : SU(n) \times S^1 \rightarrow U(n)$ given by $\phi(A, z) = zA$ is a homeomorphism.

5. Show that the group of all unitary matrices $U(n)$ is compact and connected. Regarding $U(n-1)$ as a subgroup of $U(n)$ in a natural way, recognize the quotient space as a familiar space.

6. Show that the subgroups $SU(n)$ consisting of matrices in $U(n)$ with determinant one are connected for every $n$.

7. Suppose $G$ is a topological group and $H$ is a normal subgroup, prove that $G/H$ is Hausdorff if and only if $H$ is closed.