Chapter 9 – Workforce Planning

Introduction to Lecture

This chapter presents some applications of Operations Research models in workforce planning. Workforce planning would be more of a generic application to various domains and does not restrict itself to a single domain. Typical applications would involve nurse scheduling, workforce allocation in manufacturing systems and in projects, baggage handlers, security services, Increasingly hierarchical workforce planning problems are considered where jobs require certain skills and manpower possess skill sets that are hierarchical in nature. It is possible for a person of a superior skill level to execute jobs with lower skill levels.

We present five applications of Integer Programming models for workforce planning in this chapter. These are

1. People Allocation in Line Balancing
2. Shift scheduling for baggage handlers
3. Nurse scheduling
4. Hierarchical workforce planning
5. Workforce planning in projects

Each application has an integer programming formulation and is explained using a simple numerical illustration. In each application, we have actually given multiple formulations that model various objectives and conditions relevant to the situation modelled. These applications have been taken from existing research literature or from a practical application and have been modified and simplified to suit the specific problem under consideration.

Application 1 - People Allocation in Line Balancing

The assembly line balancing problem is one of the oldest and well researched problems in the Operations Management Literature. Among the earliest works is that of Salvenson in 1955 that appeared in Industrial Engineering journal.

An assembly line is a flow system where the assembly of a product takes place. The system contains workstations arranged in a sequence and jobs are move from station to station. At each station, certain operations are repeatedly performed. The decision problem of optimally partitioning (balancing) the assembly work among the stations with respect to some objective is known as the \textit{assembly line balancing problem} (Becker and Scholl, 2006).

The job is divided into operations or tasks and each task takes a specified amount of time. There are also precedence relationships among the tasks. Given a set of tasks and their
times, the problem is to group and allocate them to a given number of workstations such that the total time on a work station is limited to a given value. The precedence constraints should be satisfied. Ordinarily it is assumed that there is only one workstation of a given type but it is also possible to create parallel workstations.

The assembly line balancing problem also includes allocation of workforce to the workstations if there are a limited number of skilled people who carry out the tasks and if they take different times to perform the tasks. Recently the literature addresses workforce allocation problems with increasing emphasis on employing the disabled (Miralles, et al, 2008), which makes it necessary to explicitly consider different operation times for different operators.

Consider an assembly line where eight components (tasks) have to be performed to assemble a product. The precedence relationships among the components (A to H) and the time to assemble are given in Table 9.1

<table>
<thead>
<tr>
<th>Component</th>
<th>Preceding</th>
<th>Duration</th>
<th>Component</th>
<th>Preceding</th>
<th>Duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>--</td>
<td>4</td>
<td>E</td>
<td>C, D</td>
<td>3</td>
</tr>
<tr>
<td>B</td>
<td>--</td>
<td>7</td>
<td>F</td>
<td>E</td>
<td>1</td>
</tr>
<tr>
<td>C</td>
<td>A, B</td>
<td>8</td>
<td>G</td>
<td>E</td>
<td>7</td>
</tr>
<tr>
<td>D</td>
<td>B</td>
<td>6</td>
<td>H</td>
<td>F</td>
<td>5</td>
</tr>
</tbody>
</table>

Table 9.1 – Precedence relationship among components.

The problem is to assign components (tasks) to workstations such that the precedence relationships are satisfied. The workstation times are the sum of the times of the activities assigned to the workstation. The cycle time, which is the maximum of the workstation times, is to be minimized. The possible number of workstations can be 2 or 3. We consider a situation where there are five employees (E1 to E5). Each station should also be allotted a person to carry out the tasks. It is initially assumed that all the employees are capable to doing all the tasks with the same efficiency and duration. It is also possible to reduce the cycle time further by considering parallel workstations.

The organization is also committed to providing employment to physically disabled people and the five employees for the assembly line activity have been recruited under this scheme. They have different operation times. These are given in Table 9.2

<table>
<thead>
<tr>
<th>Component (Task)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>4</td>
<td>5</td>
<td>3</td>
<td>5</td>
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<td>B</td>
<td>6</td>
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<td>6</td>
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<td>C</td>
<td>9</td>
<td>7</td>
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<td>6</td>
<td>9</td>
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<tr>
<td>D</td>
<td>5</td>
<td>6</td>
<td>7</td>
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<td>5</td>
</tr>
<tr>
<td>E</td>
<td>3</td>
<td>2</td>
<td>5</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>F</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Solve the problem of simultaneous assignment of components and people to workstations. Also consider parallel workstations.

After some experience with the new employees it was observed that operator 4 was finding it difficult to execute task D and operator 2 was not able to execute task A. Operator 3 had difficulty in using the space allotted to workstation 3. The possibility of reducing the cycle time by having parallel stations is also to be considered.

We first formulate the Line balancing problem to allocate tasks to workstations. We consider the case of three workstations.

Formulation 1

Let \( X_{ik} = 1 \) if task \( i \) is assigned to workstation \( k \)

The objective is to minimize the cycle time

\[
\text{Minimize } CT \]

Subject to

\[
\sum_{k=1}^{3} X_{ik} = 1 \text{ for every task } i
\]

\[
\sum_{i} p_i X_{ik} \leq CT
\]

\[
\sum_{k=1}^{3} k X_{ik} \leq \sum_{k=1}^{3} k X_{jk} \text{ for every pair of tasks } (i, j) \text{ where } i \text{ precedes } j
\]

\( X_{ik} = 0,1 \)

The first constraint assigns each task to one workstation. There are 8 constraints. The second set of constraints ensures that precedence is satisfied. If activity A precedes C, this constraint is

\[
X_{A1} + 2X_{A2} + 3X_{A3} \leq X_{C1} + 2X_{C2} + 3X_{C3}
\]
This constraint will ensure that if task C is assigned to station 2, A cannot be assigned to station 3. A will be assigned to station 1 or 2 and hence will precede C. There are 8 constraints.

The third constraint ensures that the sum of the processing times of the tasks assigned to a station does not exceed the cycle time. Here cycle time is the variable to be minimized. There are three constraints.

The formulation has 24 binary variables and 19 constraints.

The processing times for the eight tasks are 4, 7, 8, 6, 3, 1, 7 and 5 respectively. The optimum solution is given by $X_{A2} = X_{B1} = X_{C2} = X_{D1} = X_{E2} = X_{F3} = X_{G3} = X_{H3} = 1$ with $CT = 15$. Station 1 has tasks B and D (time = 13); Station 2 has tasks A, C, E (time = 15) and station 3 has F, G and H (time = 13). The cycle time is the maximum of the three times, which is 15.

Given a cycle time, we can minimize the number of work stations. The sum of processing times is 41 and if we assume $CT = 25$, it is obvious that we require a minimum of 2 workstations. We verify whether we have a feasible solution with $CT \leq 25$ and two workstations. We consider three workstations and formulate the problem.

Let $S_k = 1$ if workstation $k$ is chosen. The objective function is to

Minimize $\sum_{k=1}^{3} S_k$

The constraint

$\sum_i p_iX_{ij} \leq CT$ now becomes $\sum_i p_iX_{ij} \leq CT \times S_k$. Here CT is given. The precedence constraints and the constraints that assign each task to a workstation are retained. The formulation has 27 variables and 19 constraints.

The optimum solution to the IP is given by $X_{A2} = X_{B2} = X_{C2} = X_{D2} = X_{E3} = X_{F3} = X_{G3} = X_{H3} = 1$ with the two station times 25 and 16 respectively. Station 1 is not chosen and stations 2 and 3 are chosen. This shows that a solution with two stations is possible. Tasks A, B, C and D are assigned to the first workstation and activities E, F, G and H go to the second station. Even if we had assumed two workstations and formulated the problem we would have got a feasible solution. The number of constraints and variables would have reduced.

We can now solve the problem to minimize $CT$ for 2 workstations. The optimum solution is given by $X_{A1} = X_{B1} = X_{C1} = X_{D2} = X_{E2} = X_{F2} = X_{G2} = X_{H2} = 1$ with minimum cycle time of 22 (station 2).

Formulation 2
We now formulate the line balancing problem considering worker allocation. We consider three workstations and minimize cycle time considering different processing times for different workers.

We define $X_{iks} = 1$ if task $i$ is assigned to station $k$ which is assigned to operator $s$. We also define $Y_{ks} = 1$ if station $k$ is given to operator $s$.

The objective is to Minimize $CT$

Subject to

$$\sum_k \sum_s X_{iks} = 1 \quad \forall \ k, s$$

This constraint ensures that each task is assigned to one workstation and one operator. There are 8 constraints.

The next set of constraints ensures that an operator is assigned to a maximum of one workstation. This is given by

$$\sum_{k=1}^{3} Y_{ks} \leq 1 \quad \forall \ s.$$ There are 5 constraints. These ensure that an operator is not assigned to a station or is assigned to only one station.

The next set of constraints ensures that a workstation has exactly one operator assigned to it. This is given by

$$\sum_{s=1}^{5} Y_{ks} = 1 \quad \forall \ k.$$ There are 3 constraints.

We next write constraints for precedence relationships. This is given by

$$\sum_{k=1}^{3} \sum_{s=1}^{5} k \cdot X_{iks} \leq \sum_{k=1}^{3} \sum_{s=1}^{5} k \cdot X_{jks}.$$ There are 10 constraints – one for each precedence relation.

We now relate the station times to the cycle time. This is given by

$$\sum_i p_{is} X_{iks} \leq CT \quad \forall \ k, s.$$ There are 15 constraints.

We relate the assignment of tasks to the assignment of operators. This is given by

$$\sum_i X_{iks} \leq Y_{ks} \quad \forall \ k, s.$$ There are 15 constraints.

$X_{iks}, Y_{ks} = 0, 1.$

The formulation has 120 $X_{iks}$ variables and 15 $Y_{ks}$ variables. There are 136 variables and 66 constraints.
The optimum solution to this IP is given by $X_{A22} = X_{B14} = X_{C22} = X_{D14} = X_{E22} = X_{F33} = X_{G33} = X_{H33} = 1$; $Y_{22} = Y_{33} = Y_{41} = 1$ with $CT = 14$. Operators 4, 2 and 3 are assigned to the three workstations. Tasks B and D are assigned to workstation 1 (Operator 4 takes 13 minutes). Tasks A, C and E are assigned to station 2 (Operator 2 takes 14 minutes) and tasks F, G and H are assigned to station 3 (Operator 3 takes 13 minutes). The cycle time is 14 units.

Formulation 3
We now include the additional constraints that Operator 4 cannot do task D and Operator 2 cannot do task A. Operator 3 does not want to work in station 3.

When we include the additional constraints, we delete some of the variables from the previous formulation. Variables $X_{D14}$, $X_{D24}$ and $X_{D34}$ are left out. We also leave out $X_{A12}$, $X_{A22}$ and $X_{A32}$. Since operator 3 cannot be assigned to station 3, we leave out variables $X_{A33}$, $X_{B33}$, $X_{C33}$, $X_{D33}$, $X_{E33}$, $X_{F33}$, $X_{G33}$, $X_{H33}$, $Y_{33}$ are left out. Two constraints that are made up of entirely these variables are left out. The rest of the formulation remains.

The optimum solution to this formulation is given by $X_{A15} = X_{B15} = X_{C24} = X_{D15} = X_{E24} = X_{F24} = X_{G32} = X_{H32} = 1$; $Y_{51} = Y_{42} = Y_{23} = 1$ with $CT = 15$. Operators 5, 4 and 2 are assigned to the three workstations. Tasks A, B and D are assigned to workstation 1 (Operator 5 takes 15 minutes). Tasks C, E and F are assigned to station 2 (Operator 4 takes 11 minutes) and tasks G and H are assigned to station 3 (Operator 2 takes 12 minutes). The cycle time is 15 units.

Formulation 4
Here, we attempt to reduce the cycle time further by creating parallel workstations. We modify formulation 1 to include parallel stations. We consider the case of one parallel station. Now, the station time can be 2 times CT because of the parallel station.

Let $Z_k = 0$ if a parallel station is created in station k ($= 1$, if not created)

$$\sum_{k=1}^{3} Z_k = 2$$

We have $CT_k \leq 2CT$ and $CT_k \leq CT + M(1 - Z_k)$.

When we create a parallel station in k, $Z_k = 0$. The constraints become $CT_k \leq 2CT$ and $CT_k \leq M$ (where M is large and positive). Together we will have $CT_k \leq 2CT$. When there is no parallel station, $Z_k = 0$ and the constraint is $CT_k \leq CT$. The constraint $Z_1 + Z_2 + Z_3 = 2$ will fix only one $Z_k$ to zero and only one station has a parallel station. We have added 7 more constraints.
The optimum solution is given by \( X_{A1} = X_{B1} = X_{C2} = X_{D3} = X_{E3} = X_{F3} = X_{G3} = X_{H3} = 1; \) \( Z_1 = Z_2 = 1 \) with \( CT = 11 \). Station 1 has tasks A and B (time = 11); Station 2 has task C (time = 8) and station 3 (\( Z_3 = 0 \) indicates a parallel station) has D, E, F, G and H (time = 22). Since there are two stations, the effective time is \( 22/2 = 11 \). The cycle time is the maximum of the three times, which is \( 11 \).

The constraint \( Z_1 + Z_2 + Z_3 = 1 \) would create parallel stations in two out of the three workstations. The optimum solution now becomes \( X_{A1} = X_{B1} = X_{C2} = X_{D1} = X_{E2} = X_{F2} = X_{G3} = X_{H2} = 1; \) \( Z_1 = Z_2 = 0 \) with \( CT = 8.5 \). Station 1 has tasks A, B and D (time = 17); Station 2 has task C, E, F and H (time = 17) and station 3 has G (time = 7). Since there are two stations, the effective time in stations 1 and 2 is \( 17/2 = 8.5 \). The cycle time is the maximum of the three times, which is \( 8.5 \).

**Formulation 5**

Here we modify Formulation 2 to include parallel stations. The operator assignment constraints are modified to

\[
1 \leq \sum_{s=1}^{5} Y_{ks} \leq 2 \quad \forall \; k \quad \text{and} \quad \sum_k \sum_s Y_{ks} = 4
\]

Another set of constraints is modified to

\[
\sum_i p_{is}X_{iks} \leq CT_{ks} \quad \forall \; k, s. \; \text{And we add} \; CT \geq CT_{ks} \text{ for} \; \forall \; k, s
\]

These modified constraints assign one or two operators to each station. Since the number of operators used is 4, one station gets two operators. The cycle time for each operator is now calculated and the product cycle time is the maximum of the individual operator cycle times.

The optimum solution to this formulation is given by \( X_{A11} = X_{B11} = X_{C34} = X_{D25} = X_{E34} = X_{F33} = X_{G33} = X_{H25} = 1; \) \( Y_{11} = Y_{25} = Y_{33} = Y_{34} = 1; \) with \( CT = 10 \). Operators 1 and 5 are assigned to stations 1 and 2 while operators 3 and 4 are assigned workstation 3 (parallel stations). Tasks A and B are assigned to workstation 1 (Operator 1 takes 10 minutes). Tasks D and H are assigned to station 2 (Operator 5 takes 10 minutes) and tasks C, E, F and G are assigned to station 3. Operator 3 takes 8 minutes to do F and G and operator 4 takes 10 minutes to do C and E. The cycle time is \( 10 \) units.

To create parallel stations in two out of the three workstations, we modify only one constraint. The modified constraint is \( \sum_k \sum_s Y_{ks} = 5 \). Since the number of operators used is 5, two stations get two operators.

The optimum solution to this formulation is given by \( X_{A21} = X_{B15} = X_{C23} = X_{D21} = X_{E32} = X_{F34} = X_{G32} = X_{H34} = 1; \) \( Y_{21} = Y_{15} = Y_{23} = Y_{34} = Y_{32} = 1; \) with \( CT = 9 \). Operator 5 is assigned to station 1.
and does task B (6 minutes). Operators 1 and 3 are assigned to workstation 2 (parallel stations). Tasks A and D are assigned to operator 1 who takes 9 minutes and task C is assigned to operator 3 who takes 8 minutes. Operators 2 and 4 are assigned to workstation 3 (parallel stations). Tasks E and G are assigned to operator 2 who takes 8 minutes and tasks F and H are assigned to operator 4 who takes 7 minutes. The cycle time is 9 units.

**Formulation 6**
Here we modify formulation 5 to include additional constraints that Operator 4 cannot do task D and Operator 2 cannot do task A. Operator 3 does not want to work in station 3. We leave out the corresponding variables and constraints. Variables $X_{D14}, X_{D24}$ and $X_{D34}$ are left out. We also leave out $X_{A12}, X_{A22}$ and $X_{A32}$. Since operator 3 cannot be assigned to station 3, we leave out variables $X_{A33}, X_{B33}, X_{C33}, X_{D33}, X_{E33}, X_{F33}, X_{G33}, X_{H33}, Y_{33}$ are left out. Two constraints that are made up of entirely these variables are left out. The rest of the formulation remains.

The optimum solution for four employees is given by $X_{A11} = X_{B11} = X_{C24} = X_{D35} = X_{E32} = X_{F32} = X_{G32} = X_{H35} = 1$; $Y_{11} = Y_{24} = Y_{32} = Y_{35} = 1$; with $CT = 10$. Operators 1 and 4 are assigned to stations 1 and 2 while operators 2 and 5 are assigned workstation 3 (parallel stations). Tasks A and B are assigned to workstation 1 (Operator 1 takes 10 minutes). Task C is assigned to station 2 (Operator 1 takes 10 minutes) and tasks D, E, F G and H are assigned to station 3. Operator 2 takes 10 minutes to do E, F and G and operator 5 takes 10 minutes to do D and H. The cycle time is 10 units.

**Application 2 - Shift scheduling for baggage handlers**
Workforce scheduling is the most frequently used application of Linear Programming. Almost every text book in Operations Research or Linear Programming has an exercise or an example related to cyclic workforce scheduling. This problem is also used to explain that the corresponding LP formulation has integer solutions if the requirements are integers. An important research survey on the workforce allocation problem in cyclical scheduling is by Baker (1976).

In practical situations, each worker or operator gives 40 hours a week which is usually an 8 hour shift for 5 days. This means that all workers do not work on all days and this has to be modelled as an Integer Programming problem if there are further constraints. Baker and Magazine (1977) discuss the workforce scheduling problem with cyclic demands and day off constraints. An additional input is that the demand during the week days would be different from the demand during weekends and this is to be incorporated into the data.

With increasing emphasis on logistics management, problems such as nurse scheduling, baggage handlers, immigration officials, security services etc have gained importance with
each situation requiring constraints specific to the application area. We present an application considering baggage handlers. This can be used in many similar situations.

The number of baggage handlers required by an airline in an international airport is given by

\[4, 2, 4, 3, 4, 5, 6, 6, 7, 10, 12, 12, 10, 12, 14, 15, 16, 20, 16, 10, 8, 4\]

for each of the 24 hours starting from 12 midnight. Find the minimum number of baggage handlers to be engaged to meet the requirement. The airline is presently working three shifts (8 hours each) starting at 12 midnight, 8 am and 4 pm. They are also considering 4 am, 12 noon and 8 pm as starting times of shifts. They are also open to other possibilities. It is also necessary to consider that a person works for 40 hours in a week.

In an earlier formulation we have considered that a person works all seven days. If we consider the six given start times of shifts (12 midnight, 4 am, 8 am, 12 noon, 4 pm and 8 pm) we require 36 people. The optimum solution found using an earlier formulation is \(X_1 = 4, X_2 = 4, X_3 = 8, X_4 = 4, X_5 = 16\). This means that 4 people start work at 12 midnight, 4 people at 4 am, 8 at 8 am, 4 at 12 noon and 16 at 4 pm.

In another formulation, we assumed that a person can start work at any hour and continue to work for 8 hours. The optimum solution is given by \(X_3 = 3, X_4 = 6, X_5 = 6, X_{12} = 4, X_{13} = 6, X_{14} = 2, X_{15} = 4, X_{19} = 4\) with 35 people.

We now consider the restriction where an employee works only for 40 hours in a week. We consider three shifts starting at 12 midnight, 8 am and 4 pm. Each person works only in that shift in the week but works for only 5 consecutive days. This results in three independent problems – one for each shift. We formulate for the 12 midnight shift.

Let \(X_j\) be the number of employees starting work on day \(j\). The constraints for the seven days are

\[
\begin{align*}
X_4 + X_5 + X_6 + X_7 + X_1 & \geq 6 \\
X_5 + X_6 + X_7 + X_1 + X_2 & \geq 6 \\
X_6 + X_7 + X_1 + X_2 + X_3 & \geq 6 \\
X_7 + X_1 + X_2 + X_3 + X_4 & \geq 6 \\
X_1 + X_2 + X_3 + X_4 + X_5 & \geq 6 \\
X_2 + X_3 + X_4 + X_5 + X_6 & \geq 6 \\
X_3 + X_4 + X_5 + X_6 + X_7 & \geq 6 
\end{align*}
\]

The RHS value is 6 which is the maximum requirement for the first eight hours. The demand for Monday would be met by people who start work on Monday (\(X_1\)), Thursday (\(X_4\)) Friday (\(X_5\)), Saturday (\(X_6\)) and Sunday (\(X_7\)) because each person works for 5 consecutive days. The objective function is to minimize the total number of workers and is given by
Minimize $X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7$ and the non-negativity constraints $X_j \geq 0$. The optimum solution to the LP is given by

$X_1 = X_2 = X_3 = X_4 = X_6 = 1; \ X_5 = X_7 = 2$ with 9 baggage handlers. Several solutions exist that require 9 people. For example, $X_2 = X_3 = X_4 = X_5 = X_7 = 1; \ X_1 = X_6 = 2$ is also optimal. The formulation for the second shift is the same with requirement $= 12$. The optimum solution is $X_2 = X_3 = X_5 = X_7 = 2; \ X_1 = X_4 = X_6 = 3$ with 17 baggage handlers. The formulation for the third shift is also the same with requirement $= 20$. The optimum solution is $X_1 = X_2 = X_3 = X_4 = X_5 = X_6 = X_7 = 4$ with 28 baggage handlers. A total of 54 baggage handlers are required if we include the condition that an employee works for 5 shifts (or 40 hours) in a week. We observe that the requirement goes up by 50% in this case.

In the previous formulation, we have assumed that the requirement is same on all days of the week. Considering the first shift, we have a total demand of 42 people. This is redistributed to $5 \times 5 + 7 + 8$ (for the weekend). Similarly the total demand of 84 in second shifts is distributed as $5 \times 10 + 2 \times 17$ and the total demand of 140 is distributed as $5 \times 18 + 2 \times 25$. The formulations are solved again with the new values of the RHS.

The optimum solution to the first shift problem indicates that 9 people are required. The solution is $X_3 = X_5 = 4; \ X_7 = 1$. We observe that though the number of people to be employed is the same, more people come to work in fewer starting slots. This is because of the variation caused by the increase in demand during the weekend. The optimum solutions to the other two problems indicate that 18 and 29 people are required resulting in a total of 56 people. These solutions also show similar behaviour in terms of more people coming to work in fewer slots.

We consider the first variation where we look at three shifts but relax the assumption that a person works for five consecutive days. We now assume that a person works in the same shift throughout the week but can work for any five days. Each employee does not work on two days.

Let $X_{ij}$ be the number of people who do not work on days $i$ and $j$. There are 21 variables since there are seven days and $^7C_2$ ways of choosing the two days of leave. If we consider the first shift problem and day 1 (Monday), we have the constraint

$$X_{23} + X_{24} + X_{25} + X_{26} + X_{27} + X_{34} + X_{35} + X_{36} + X_{37} + X_{45} + X_{46} + X_{47} + X_{56} + X_{57} + X_{67} \geq 6$$

In general, for day $k$, we have the constraint $\sum_{i=1}^{6} \sum_{j=i+1}^{7} X_{ij} \geq 6$. There are seven constraints. The objective function is to Minimize $\sum_{i=1}^{6} \sum_{j=i+1}^{7} X_{ij}$, $X_{ij} \geq 0$ and integer.
The optimum solution considering demand = 6 on all days in first shift is given by \( X_{13} = X_{34} = X_{47} = X_{56} = 2 \) with 8 baggage handlers. The earlier formulation where we assumed five consecutive days, required 9 people. The second shift problem gives 16 people when the demand is 12 for all days. The third shift problem gives 27 people when the demand is 20 for all days. The total people required are 52 people as against a total of 54 when we had the five consecutive day’s assumption.

When we solve the problem with data representing variation in demand (higher demand during weekend), that we used earlier we require 9, 17 and 27 people resulting in a total of 53 people as against 56 people when we have the consecutive day assumption.

We now consider shifts that start six times a day (12 midnight, 4 am, 8 am, 12 noon, 4 pm, 8 pm). Each shift is for 8 hours from the start of the shift and each person works for 5 consecutive days in the same shift. The decision variables are

\[ X_{ij} = \text{number of people start working in a week on day } i \text{ in shift } j. \]

They work for 8 hours which means that they work for two consecutive shifts. They start work next day in the same time and work for five consecutive days. Since a person works for two consecutive shifts, we have a single formulation for all the three shifts together.

The demands for the 6 shifts are 4, 6, 12, 12, 20 and 16. If we consider meeting the demand for Monday (day 1) first shifts, the constraint will be

\[ X_{11} + X_{41} + X_{51} + X_{61} + X_{71} + X_{76} + X_{36} + X_{46} + X_{56} + X_{66} \geq 6. \]

The first five terms denote people starting work in shift 1 on Monday, Thursday, Friday, Saturday and Sunday. The next five terms represent people who start work on the previous shift (6th shift on the previous day) on Sunday, Wednesday, Thursday, Friday and Saturday. In general, the constraint is

\[ \sum_{i=k-4}^{i} \sum_{j=j-1}^{j} X_{kl} \geq d_{ij}. \]

In these constraints, we have to add 7 when \( k=1 \) to 4 and 6 when \( j = 7, 1 \) There are 21 constraints and 42 variables. Each constraint has 10 variables.

The objective is to Minimize \( \sum_{i=1}^{7} \sum_{j=1}^{6} X_{ij} \). We also have \( X_{ij} \geq 0 \) and integer.

The optimum solution is given by \( X_{12} = X_{22} = X_{23} = X_{26} = X_{31} = X_{42} = X_{44} = X_{51} = X_{52} = X_{56} = X_{62} = X_{64} = 1; X_{13} = X_{24} = X_{33} = X_{43} = X_{53} = X_{63} = X_{72} = X_{74} = X_{75} = X_{76} = 2; X_{56} = 3, X_{15} = X_{35} = X_{45} = X_{65} = 4 \) with 54 people.

We also solve for the data that showed more variation with increase in demand during weekend. The optimum solution gave the requirement as 56 people. For the given data, we observe that there is no advantage in considering six different starting shifts. The simpler
formulations with three shifts gave the same solution. However, when we relaxed the five consecutive day assumption, fewer people are required.

We now consider the six starting shifts in a day (12 midnight, 4 am, 8 am, 12 noon, 4 pm, 8 pm). Each shift is for 8 hours from the start of the shift and each person works for 5 days in a week in the same shift. We have relaxed the five consecutive day assumption here. The decision variable would be

\[ X_{ijk} = \text{number of people who do work on all days of the week other than i and j and start work in shift } k; \text{ } = 0 \text{ otherwise.} \]

There are 21 x 6 = 126 variables because 5 days in a week out of 7 can be chosen in \( \binom{5}{2} = 21 \) ways and there are 6 shifts in each day. The objective is to Minimize \( \sum_{i,j=1}^{21} \sum_{k=1}^{6} X_{ijk} \)

The constraint set is \( \sum_{i,j \neq l}^{21} X_{ijk} + \sum_{i,j \neq l}^{21} X_{ij,k-1} \geq d_{lk} \)

There are 42 constraints and there are 30 terms in each constraint. The constraint to meet the demand on Monday first shift (\( d_{11} \)) is given by

\[ X_{231} + X_{241} + X_{251} + X_{261} + X_{271} + X_{341} + X_{351} + X_{361} + X_{371} + X_{451} + X_{461} + X_{471} + X_{561} + X_{571} + X_{671} + X_{126} + X_{136} + X_{146} + X_{156} + X_{166} + X_{236} + X_{246} + X_{256} + X_{266} + X_{346} + X_{356} + X_{366} + X_{456} + X_{466} + X_{566} \geq d_{11}. \]

All \( X_{ij1} \) where \( i, j \neq 1 \) and \( k = 1 \) represent people who work on Monday in shift 1. All \( X_{ij6} \) where \( i, j \neq 6 \) and \( k = 6 \) work on Sunday 6th shift and therefore work on Monday first shift.

We use the data \( d_{i1} = 4, d_{i2} = 6, d_{i3} = 12, d_{i4} = 12, d_{i5} = 20 \) and \( d_{i6} = 16 \) for all 7 days. The optimum solution requires 51 people. We also solve for the data that showed more variation with increase in demand during weekend. The optimum solution gave the requirement as 54 people.

**Application 3 - Nurse Scheduling**

The nurse scheduling problem is to assign nurses to shifts and work days in a hospital to meet the requirements related to patient care. It is also necessary to minimize underutilization so that costs can be reduced. Sometimes it may be required to have the same nurse attend specific patients so that service to the patient can be maximized. This problem has several constraints and sometimes leads to multiple objective optimization problems.

The nurse scheduling problem has been addressed extensively in the literature. An early paper is by Arthur and Ravindran (1981) where they develop a multiple objective solution to
this problem. The model that we formulate here is an adaptation of the model proposed by Azaiez and Sharif (2005), who proposed a goal programming formulation. The model proposed in this application is a binary IP model.

For our problem instance we assume that there are two shifts – a day shift and a night shift. We consider nine consecutive days. The requirement of nurses in day shift is 3 and night shift is 1. A nurse, if working on a day is assigned to only one shift. A nurse working on night shift in one day cannot work on day shift the next day. A nurse can work for a maximum of 4 consecutive days. Days 6 and 7 are weekend days and every nurse should be given at least one day holiday during the weekend. At present eight nurses are available. Can we find a feasible allocation? The hospital desires that the number of nurses doing night shift be small while some nurses feel that night shift should be applicable to all.

On each of the nine days a nurse is either working in day shift or working in night shift or is not working. Let \( XD_{ik} = 1 \) if nurse \( i \) works in day shift in day \( k \); = 0 otherwise. Let \( XN_{ik} = 1 \) if nurse \( i \) works in night shift in day \( k \); = 0 otherwise. Let \( XL_{ik} = 1 \) if nurse \( i \) is given day off on day \( k \); = 0 otherwise.

We have \( i = 1,..., 8 \) (eight nurses) and \( k = 1,..., 9 \) (nine days). We have 72 variables for \( XD_{ik}, XN_{ik} \) and \( XR_{ik} \) resulting in 216 variables.

The demand for nurses for day and night shifts has to be met.

\[
\sum_{i=1}^{8} XD_{ik} \geq 3 \quad \text{for } k = 1 \text{ to } 9 \quad \text{and}
\]

\[
\sum_{i=1}^{8} XN_{ik} \geq 1 \quad \text{for } k = 1 \text{ to } 9
\]

On any given day, each nurse either works in day shift or in night shift or has a day off. This is given by

\[
XD_{ik} + XN_{ik} + XR_{ik} = 1 \quad \text{for } i, k. \text{ There are 72 constraints.}
\]

A nurse does not work in a day shift after working in the night shift the previous day.

\[
XD_{i+1,k} + XN_{ik} \leq 1 \quad \text{for } i = 1 \text{ to } 7, k = 1 \text{ to } 9. \text{ There are 63 constraints}
\]

A nurse cannot come to work for more than 3 consecutive days. This means that a nurse should have at least one day off for any five consecutive days.
\[ XR_{i,k} + XR_{i,k+1} + XR_{i,k+2} + XR_{i,k+3} \geq 1 \text{ for } i = 1 \text{ to } 8, k = 1 \text{ to } 6. \] There are 48 constraints.

The objective is to check if we have a feasible solution with 8 nurses. Our objective function is to Minimize \( \sum_{k=1}^{9} XD_{8k} + XN_{8,k} \).

The formulation has 216 binary variables and 201 constraints. The optimum solution is given by \( XD_{11} = XD_{21} = XD_{71} = XD_{12} = XD_{13} = XD_{34} = XD_{43} = XD_{45} = XD_{55} = XD_{75} = XD_{16} = XD_{66} = XD_{17} = XD_{57} = XD_{87} = XD_{38} = XD_{48} = XD_{68} = XD_{39} = XD_{49} = XD_{69} = 1; XN_{31} = XN_{42} = XN_{23} = XN_{14} = XN_{25} = XN_{26} = XN_{77} = XN_{78} = XN_{79} = 1. \) The rest are \( XR_{ik} \) variables and all are equal to 0.

The solution is shown in Table 9.3

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Table 9.3 – Solution to the nurse scheduling problem

The optimum solution gives the objective function as 1 with \( X_{87} = 1 \). This means that we require eight nurses where one of the nurses is used only for 1 day on day 7.

If we change the requirement of nurses on day shift on day 7 to 2 nurses, we have the same solution with \( X_{87} = 0 \) and \( Z = 0 \) indicating that seven nurses are enough.

We have now found a feasible solution with 8 nurses. We now consider some goals that are going to be modelled as flexible constraints. We now add a constraint that minimizes the number of nurses working in night shift.

We introduce a variable \( Y_i = 1 \) if nurse \( i \) works in the night shift on any of the days. There are 8 more binary variables. We now add the constraint

\[ \sum_{j=1}^{9} XN_{ij} \leq MY_i \forall i. \] There are 8 constraints. The objective function now Minimizes \( \sum_{i=1}^{8} Y_i \).

The formulation now has 224 binary variables and 209 constraints. The optimum solution is given by \( XD_{11} = XD_{41} = XD_{51} = XD_{12} = XD_{32} = XD_{42} = XD_{52} = XD_{62} = XD_{33} = XD_{63} = XD_{73} = XD_{83} \).
We have two nurses who work in the night shifts. In this solution we observe that more nurses work in day shifts than in the previous solution. This is because the objective function is not concerned about the total number of day shift nurses. If we wish to keep the total number of nurses in day shifts to 27 and maintain the daily requirement we should add a constraint \( \sum_{i=1}^{9} \sum_{j=1}^{8} X_{D_{ij}} = 27 \). The model would resemble a goal programming formulation where we keep previous solutions with objectives and goals intact and optimize further to meet additional goals.

(In the previous solution, we actually had 28 day shifts over 9 days because one day had 4 nurses. This again was due to the fact that the objective function did not \( \text{Minimize} \sum_{i=1}^{9} \sum_{j=1}^{8} X_{D_{ij}} \). The optimum solution after the addition of the constraint is given in Table 9.4. We observe that day 8 has two people working in night shift. This has happened because we have not restricted the total night shift work to 9 days. We can leave out one of them for the night shift on day 8.

We now keep the number of people working in the night shift to two by adding the constraint \( \sum_{i=1}^{8} Y_{i} = 2 \). We now try to model another goal where each nurse spends more days in a day shift than in night shift. We now try to find a feasible solution where \( \sum_{k=1}^{9} X_{D_{ik}} \geq \sum_{k=1}^{9} X_{N_{ik}} + 1 \) \( \forall i \) is satisfied. This results in eight additional constraints. For nurse 1, this constraint would be

\[ XD_{11} + XD_{12} + XD_{13} + XD_{14} + XD_{15} + XD_{16} + XD_{17} + XD_{18} + XD_{19} \geq XN_{11} + XN_{12} + XN_{13} + XN_{14} + XN_{15} + XN_{16} + XN_{17} + XN_{18} + XN_{19} + 1. \]

In order to see whether we can meet the goal where each nurse works for more days in day shift than in night shift, we use goal programming where we write the constraint \( \sum_{k=1}^{9} X_{D_{ik}} \geq \sum_{k=1}^{9} X_{N_{ik}} + 1 \) \( \forall i \) as \( \sum_{k=1}^{9} X_{D_{ik}} - \sum_{k=1}^{9} X_{N_{ik}} + p_{i} - q_{i} = 1 \) \( \forall i \) where \( p_{i} \) and

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Table 9.4 – Optimum solution after adding constraint.
q$_i$ are deviation variables and we minimize $\sum_{i=1}^{8} p_i$. If there is a feasible solution where the additional constraint is satisfied, we will have $\sum_{i=1}^{8} p_i = 0$. Else we will have a positive value for $\sum_{i=1}^{8} p_i$.

We also fix the condition that we have a total of 8 nurses and only 2 can work in night shift. We also fix that the total number of nurse day shifts is 27 so that all earlier solutions and conditions are not violated.

The optimum solution is given in Table 9.5

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Table 9.5 – Optimum solution with additional restriction

This solution also has $Y_1 = Y_5 = 1$ indicating that nurses 1 and 5 work in night shifts. We also have $p_1 = 4$, $p_5 = 2$, $q_3 = 2$, $q_4 = 2$, $q_6 = 3$, $q_7 = 4$, $q_8 = 5$. This shows a deviation of 6 in the objective function from the desired value of zero. Both the nurses 1 and 5 work more night shifts than day shifts. This shows that if we have only 2 nurses working in night shifts the goal cannot be achieved.

In fact, the earlier solution in Table 9.6 also shows both the nurses working more days in night shifts than day shifts.

If we wish to achieve the goal, we have to relax the number of nurses working in night shifts. We now solve for 3 nurses working in night shift. The optimum solution is given in Table 9.6

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Table 9.6 – Optimum solution after relaxing number of nurses working in night shift.

We observe that we have a solution where three nurses work in night shifts and every nurse works more days in day shifts than in night shifts. We also observe that nurse 3, 4 and 7 work for fewer days than nurse 2, 5 and 8 who are overloaded. We may offset this through a roster where the nurses 2 and 3 shift positions in the coming week.

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