**TSP with time windows**

In many practical situations, the TSP problem has additional constraints of time windows. A single salesman visiting multiple customers can be asked to report to a customer after a certain time. Alternately, the salesman may be asked to visit before a certain time or between two time limits. When there is a restriction to visit a place within a time window, the problem becomes a TSP with time windows. The formulation becomes different because the time of visiting a place becomes a variable.

We assume that the salesman leaves city 1, visits all other cities once and only once and returns to city 1. The TSP circuit problem becomes a path problem where the path starts and ends at city 1. All time windows are relative to city 1, which the salesman starts at time zero.

Let \( t_j \) be time at which the salesman (starts from) or reaches city \( j \). If there are \( n \) cities, we assume that city 1 becomes city \( n+1 \) (which is the final destination). The formulation from Black (1983) is considered for illustration.

Minimize \( t_{n+1} - t_1 \)

Subject to

\[
\begin{align*}
  t_{j} - t_1 & \geq d_{1j} \quad j = 2, \ldots, n \\
  t_{n+1} - t_j & \geq d_{1j} \quad j = 2, \ldots, n \\
  \left| t_i - t_j \right| & \geq d_{ij} \text{ for } i = 3, \ldots, n; \ j < i \\
  t_j & \geq 0
\end{align*}
\]

The explanation to the formulation is as follows: As mentioned city 1 is the starting city and city \( n+1 \) which is actually city 1 is the destination. It is assumed that the salesman visits each city once and only once but spends negligible time in the city. The objective function minimizes the difference between \( t_{n+1} \) and \( t_1 \) which essentially minimizes the time at which the salesman returns to city 1.

The first set of constraints ensures that the salesman visits exactly one city from city 1 (cities 2 to \( n \)). The second set of constraints ensures that the salesman comes back to 1 (now called city \( n+1 \)) from one out of cities 2 to \( n \). The third set of constraints ensures that the salesman visits cities 2 to \( n \) in any order. The absolute value term ensures the condition. The time of arrival in a city enables that we minimize the time taken to come back to city 1. The term \( d_{ij} \) represents the time taken to go from city \( i \) to \( j \).

The formulation as such cannot be fed to a solver because of the absolute value term. It looks like an LP formulation but it is not. The term \( \left| t_i - t_j \right| \geq d_{ij} \) has to be linearized to
convert it to a formulation that can go to a solver. For each of the constraint, we introduce two constraints and a binary variable. The constraint \( |t_{ij} - t_{ij}| \geq d_{ij} \) is replaced by \( d_{ij} - (t_i - t_j) \leq M\delta_{ij} \) and \( d_{ij} - (t_j - t_i) \leq M(1-\delta_{ij}) \).

This leads us to a binary IP with \( 2n + \binom{n}{2} \) constraints.

The above formulation has not considered time windows and can be used as a formulation for a TSP. Additional constraint such as

\[ l_j \leq t_j \leq u_j \]

can be added when there are time windows.

This formulation assumes that the \( d_{ij} \) values follow triangle inequality. For example, if a particular \( d_{ij} \) value (where \( i \neq j \)) has a value of infinity, it can force one of the \( t_j \) values to a large value. We will show this through example

**Illustration 8.5**

Consider the data given in Table 8.3. Assume that the values represent the time taken to travel from city \( i \) to \( j \). Formulate the TSP using the formulation where the time to reach a city is a decision variable and solve. What happens to the optimum solution when the salesman has to visit city before 20, city 3 before 30 and city 4 after 50?

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We introduce variables \( t_1 \) to \( t_6 \) in the formulation where city 6 represents city 1. The salesman returns to city 6 (which is actually city 1). The formulation is to

\[
\text{Minimize } t_6 - t_1 \\
\text{subject to } \\
t_2 - t_1 \geq 12 \\
t_3 - t_1 \geq 8 \\
t_4 - t_1 \geq 16 \\
t_5 - t_1 \geq 9 \\
t_6 - t_1 \geq 12 \\
t_6 - t_2 \geq 8 \\
t_6 - t_4 \geq 16
\]
\[ t_6 - t_2 \geq 9 \]
\[ t_3 - t_2 + 1000d_{23} \leq 982 \]
\[ t_2 - t_3 - 1000d_{23} \leq -18 \]
\[ t_4 - t_2 + 1000d_{24} \leq 989 \]
\[ t_2 - t_4 - 1000d_{24} \leq -11 \]
\[ t_4 - t_3 + 1000d_{34} \leq 990 \]
\[ t_3 - t_4 - 1000d_{34} \leq -10 \]
\[ t_5 - t_2 + 1000d_{25} \leq 993 \]
\[ t_2 - t_5 - 1000d_{25} \leq -7 \]
\[ t_5 - t_3 + 1000d_{35} \leq 992 \]
\[ t_3 - t_5 - 1000d_{35} \leq -8 \]
\[ t_5 - t_4 + 1000d_{45} \leq 991 \]
\[ t_4 - t_5 - 1000d_{45} \leq -9 \]

This formulation has 6 continuous variables and 6 binary variables. It has 20 constraints. The formulation in example 1 has 20 variables and 20 constraints. All variables are binary.

The optimum solution to the above formulation is \( t_1 = 0, t_2 = 16, t_3 = 34, t_4 = 27, t_5 = 9 \) and \( t_6 = 45 \). Sorting the \( t_j \) values in increasing value, the optimum tour is 1-5-2-4-3-1 with time = 45.

We add the three constraints representing the time windows. These are \( t_2 \leq 20, t_3 \leq 30 \) and \( t_4 \geq 50 \).

The optimum solution is \( t_1 = 0, t_2 = 12, t_3 = 30, t_4 = 50, t_5 = 19 \) and \( t_6 = 66 \). Sorting the \( t_j \) values in increasing value, the optimum tour is 1-2-5-3-4-1 with time = 66. The salesman starts from city 1 at \( t = 0 \), goes to city 2 at \( t = 12 \), goes to city 5 at \( t = 19 \), reaches city 3 at \( t = 30 \) and reaches city 4 at \( t = 40 \). The delivery takes place at \( t = 50 \) (which satisfies the constraint \( t_4 \geq 50 \)). The salesman leaves city 4 at 50 and comes back at 66. We may also say that the person waits in city 3 for 10 additional units and reaches \( t_4 = 50 \).

**Minimizing waiting times**

We look at one particular version of the problem where the salesman could be a doctor or could be a service mechanic. We assume that city 1 is the residence of the doctor and she has to visit 4 patients (cities 2 to 5). All the patients expect the doctor to come to their homes as quickly as possible. If we assume that the doctor spends exactly the same time with each patient, only the travel times matter and we can ignore the constant service time (in our optimization). In such cases the objective function would be to minimize the total waiting time of all the patients which is to

Minimize \( t_1 + t_2 + t_3 + \ldots + t_n \).
If the travelling salesman is a technical mechanic who has to attend faults in equipment in customer homes, the problem is similar and the same objective function can be used. This problem is called the travelling repairman problem or deliveryman problem.

**Illustration 8.6**

Consider the data in Table 8.3. Find the sequence (tour) from city 1 such that the total waiting time for customers in cities 2 to 5 is minimized?

The objective function changes to Minimize $t_1 + t_2 + t_3 + t_4 + t_5$. The constraints are the same as in Example 5. There is no time window constraint.

The optimum solution is $t_1 = 0, t_2 = 36, t_3 = 8, t_4 = 25, t_5 = 16$ and $t_6 = 48$ and objective function value = 85. Sorting the $t_i$ values in increasing value, the optimum tour is 1-3-5-4-2-1 with the salesman returning at $t = 48$. The total waiting time at the cities 2 to 5 by the customers is $36 + 8 + 25 + 16 = 85$.

We compare this solution with the solution that minimizes total time for the salesman. This solution is $t_1 = 0, t_2 = 16, t_3 = 34, t_4 = 27, t_5 = 9$ and $t_6 = 45$. Sorting the $t_i$ values in increasing value, the optimum tour is 1-5-2-4-3-1 with time = 45 for the salesman to return. The total waiting time at the customers is $16 + 34 + 27 + 9 = 86$.

**Pickup and delivery problem or Dial – a- ride problem (Cordeau and Laporte (2003))**

A very popular version of the TSP is what is called the pickup and delivery problem where the salesman visits some cities to pickup material and delivers it in other cities. The salesman may have to visit all the pickup points first and collect all the material and then deliver them to the delivery points. If the number of pickup and delivery points is equal and same quantities are to be picked up and delivered, the salesman can have two pickups followed by two deliveries and so on.

If the salesman is a taxi driver, every pickup has to be followed by a delivery because each pickup may be a passenger who has to be dropped at the destination. This particular version of the pickup and delivery problem is called the “dial-a-ride problem”. Sometimes, these problems also have associated time windows.

The formulation involving $t_i$ variables is a convenient way to model the pickup and delivery problems. We illustrate several cases using the example.
Illustration 8.7
Consider the data in Table 8.3. Assume that cities 1 and 4 are pickup points and cities 2, 3 and 5 are delivery points. Find the sequence (tour) from city 1 such that the salesman visits all the pickup points first and then the delivery points to minimize total distance travelled?

1. Assume that the demand in each city (2, 3 and 5) is \( d \). Assume that the salesman picks up 2\( d \) from city 1 and \( d \) from city 4. Find the sequence that minimizes total distance travelled.

2. Assume a dial-a-ride problem where there are 2 customers who want to go from 2-4 and 5-3. Find the tour that minimizes total distance travelled and that minimizes sum of waiting times of the two customers?

In the first case cities 1 and 4 are pickup points and cities 2, 3 and 5 are delivery points. We have to visit the pickup points first and then the delivery points. The formulation is same as in Example 5 with additional constraints to ensure that the salesman visits the pickup points first and then the delivery points. The additional constraints are

\[
\begin{align*}
t_2 & \geq t_1 + d_{12} \\
t_3 & \geq t_1 + d_{13} \\
t_5 & \geq t_1 + d_{15} \\
t_2 & \geq t_4 + d_{24} \\
t_3 & \geq t_4 + d_{34} \\
t_5 & \geq t_4 + d_{45}
\end{align*}
\]

The optimum solution is \( t_1 = 0, t_2 = 27, t_3 = 45, t_4 = 16, t_5 = 34 \) and \( t_6 = 53 \) and objective function value = 53. Sorting the \( t_j \) values in increasing value, the optimum tour is 1-4-2-5-3-1 with the salesman returning at \( t = 53 \). We observe that the salesman first visits the pickup points 1 and 4 and then the delivery points 2, 3 and 5.

In the second case, the salesman picks up 2\( d \) from city 1 and \( d \) from city 4 (the two supply points). The basic formulation is the same as in example 5 with the addition of constraints that ensure that a maximum of two cities can be reached after city 1 before reaching city 4. The constraints are

\[
\begin{align*}
t_2 & \geq t_1 + d_{12} \\
t_3 & \geq t_1 + d_{13} \\
t_5 & \geq t_1 + d_{15}
\end{align*}
\]

with the condition that a maximum of two out of the three can hold. We model this by rewriting them as

\[
\begin{align*}
t_1 + d_{12} - t_2 & \leq My_1 \\
t_1 + d_{13} - t_3 & \leq My_2 \\
t_1 + d_{14} - t_4 & \leq My_3 \\
y_1 + y_2 + y_3 & = 1 \\
y_j & = 0,1.
\end{align*}
\]
The constraint $y_1 + y_2 + y_3 = 1$ would keep one of the $y_i$ to 1 and that constraint will be redundant while the other two will be used. Therefore two out of the three constraints will be valid.

The additional constraint that maximum of one of

- $t_4 + d_{24} - t_2 \leq M y_4$
- $t_4 + d_{34} - t_3 \leq M y_5$
- $t_4 + d_{54} - t_5 \leq M y_6$
- $y_4 + y_5 + y_6 = 2$
- $y_j = 0, 1.$

The constraint $y_4 + y_5 + y_6 = 2$ would keep two of the $y_i$ to 1 and these constraint will be redundant while the other will be used. Therefore one out of the three constraints will be valid and one city out of 2, 3 and 5 will be visited after city 4.

In fact, for this particular instance, we can leave out the first three additional constraints. Since we begin with city 1, we are going to pick up 2d items. The only constraint is that one city out of 2, 3 and 5 should be visited after 4.

The optimum solution after adding the second set of constraints is given by $t_1 = 0, t_2 = 29, t_3 = 8, t_4 = 18, t_5 = 36$ and $t_6 = 45$ and objective function value = 45. Sorting the $t_j$ values in increasing value, the optimum tour is 1-3-4-2-5-1 with the salesman returning at $t = 45.$ We observe that the salesman visits the pickup point 5 after visiting 4, which is given by $y_4 = y_5 = 1.$ This makes $y_6 = 0$ making the constraint $t_5 \geq t_4 + d_{45}$ binding.

(The above formulation would involve additional constraints if we have more supply points and more demand points)

The third case involves a dial-a-ride problem with two customers wanting to go from 2 to 4 and from 5 to 3. The basic formulation is the same as in example 5. The additional constraints should ensure that salesman visits 4 immediately after 2 and visits 3 immediately after 5 because the customers go to their destination immediately on getting into the vehicle. The additional constraints are

- $t_4 = t_2 + d_{24}$
- $t_3 = t_5 + d_{35}.$

The optimum solution is given by $t_1 = 0, t_2 = 12, t_3 = 40, t_4 = 23, t_5 = 32$ and $t_6 = 48$ and objective function value = 48. Sorting the $t_j$ values in increasing value, the optimum tour is 1-2-4-5-3-1 with the salesman returning at $t = 48.$
It is observed that salesman visits city 4 immediately after 2 and visits 3 immediately from 5.

**TSP with subsets**

There are situations where the salesman visits a subset of the available cities. These situations are:

1. The salesman collects revenue in each city and returns to the starting city when a certain amount is collected. The problem is to travel minimum distance to collect the required amount. This problem is called the **orienteering problem**. The salesman can also return to a known destination.

2. Another version is to maximize the amount collected subject to a distance constraint. If the revenue in each city is 1, the problem reduces to visiting maximum number of cities and returning to the starting point subject to a distance constraint.

In addition there could a penalty associated with a city not visited and the objective function could include the penalty.

**Orienteering problem (Golden et al, 1987)**

We formulate the orienteering problem considering that the salesman returns to the stating city. The two versions of the orienteering problem are the same except that destination becomes the starting city when the salesman returns to the starting city. Different formulations of this problem are possible. Our formulation is as follows:

Let \( Y_j = 1 \) if the salesman visits city \( j \). Let \( t_j \) be the time at which the salesman visits city \( j \). Also city \( n+1 \) is the same as city 1.

Minimize \( t_{n+1} - t_1 \)

Subject to

\[
\begin{align*}
    t_j & - t_1 \geq d_{1j} Y_j \\
    t_{n+1} - t_j & \geq d_{1j} Y_j \\
    |t_j - t_i| & \geq d_{ij} Y_i Y_j \text{ for all } i,j = 2,..,n \text{ and } i \neq j.
\end{align*}
\]

\( Y_j = 0,1 \).

The above formulation is not an LP because of the absolute value constraint. The above formulation is non linear because of the product term \( Y_i Y_j \). We convert this to a zero-one IP by the addition of the following constraints for each of the absolute values.

\[
\begin{align*}
    D_{ij} Y_i + t_j - t_i & \leq M \delta_{ij} \\
    D_{ij} Y_i + t_i - t_j & \leq M(1-\delta_{ij}) \\
    Y_i & \geq Y_i + Y_j - 1
\end{align*}
\]
The introduction of the binary $\delta_{ij}$ to convert the absolute values has already been explained in Example 5. The introduction of $Y_{ij}$ converts the non linear problem into a linear problem. The two constraints relating $Y_{ij}$ to $Y_i$ and $Y_j$ ensure that $Y_{ij}$ will take the value 1 only when $Y_i = Y_j = 1$ (when both the cities are visited) and will take zero when one of the cities is not visited or when both are not visited.

**Illustration 8.8**

Consider a 6 city TSP where the distance data is given in *Table 8.4*. The revenue in cities 2 to 6 are 100, 70, 80, 60 and 90.

- a) Solve the orienteering problem where the salesman has to collect at least Rs 300.
- b) Find the maximum revenue collected if there is a distance restriction of 44 and 40?
- c) Find the minimum distance travelled to collect at least Rs 250? There is a penalty of Rs 2 for every city not visited

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Let $Y_j = 1$ if the salesman visits city $j$. Let $t_j$ be the time at which the salesman visits city $j$. Also city 7 is the same as city 1.

The formulation is to Minimize $t_7 - t_1$

Subject to

$t_2 - t_1 \geq 12$
$t_3 - t_1 \geq 8$
$t_4 - t_1 \geq 16$
$t_5 - t_1 \geq 9$
$t_6 - t_1 \geq 10$
$t_7 - t_2 \geq 12$
$t_7 - t_3 \geq 8$
$t_7 - t_4 \geq 16$
$t_7 - t_5 \geq 9$
$t_7 - t_6 \geq 10$
\[ t_3 - t_2 + 1000d_{23} + 18y_{32} \leq 1000 \]
\[ t_2 - t_3 - 1000d_{23} + 18y_{32} \leq 0 \]
\[ t_4 - t_2 + 1000d_{24} + 11y_{42} \leq 1000 \]
\[ t_2 - t_4 - 1000d_{24} + 11y_{42} \leq 0 \]
\[ t_4 - t_3 + 1000d_{34} + 10y_{43} \leq 1000 \]
\[ t_3 - t_4 - 1000d_{34} + 10y_{43} \leq 0 \]
\[ t_5 - t_2 + 1000d_{25} + 7y_{52} \leq 1000 \]
\[ t_2 - t_5 - 1000d_{25} + 7y_{52} \leq 0 \]
\[ t_5 - t_3 + 1000d_{35} + 8y_{53} \leq 1000 \]
\[ t_3 - t_5 - 1000d_{35} + 8y_{53} \leq 0 \]
\[ t_5 - t_4 + 1000d_{45} + 9y_{54} \leq 1000 \]
\[ t_4 - t_5 - 1000d_{45} + 9y_{54} \leq 0 \]
\[ t_6 - t_2 + 1000d_{26} + 8y_{62} \leq 1000 \]
\[ t_2 - t_6 - 1000d_{26} + 8y_{62} \leq 0 \]
\[ t_6 - t_3 + 1000d_{36} + 12y_{63} \leq 1000 \]
\[ t_3 - t_6 - 1000d_{36} + 12y_{63} \leq 0 \]
\[ t_6 - t_4 + 1000d_{46} + 11y_{64} \leq 1000 \]
\[ t_4 - t_6 - 1000d_{46} + 11y_{64} \leq 0 \]
\[ t_6 - t_5 + 1000d_{56} + 9y_{65} \leq 1000 \]
\[ t_5 - t_6 - 1000d_{56} + 9y_{65} \leq 0 \]
\[ y_{32} - y_{23} - y_3 \geq -1 \]
\[ 2y_{32} - y_{23} - y_3 \leq 0 \]
\[ y_{42} - y_{24} - y_4 \geq -1 \]
\[ 2y_{42} - y_{24} - y_4 \leq 0 \]
\[ y_{52} - y_{25} - y_5 \geq -1 \]
\[ 2y_{52} - y_{25} - y_5 \leq 0 \]
\[ y_{62} - y_{26} - y_6 \geq -1 \]
\[ 2y_{62} - y_{26} - y_6 \leq 0 \]
\[ y_{43} - y_{34} - y_3 \geq -1 \]
\[ 2y_{43} - y_{34} - y_3 \leq 0 \]
\[ y_{53} - y_{45} - y_5 \geq -1 \]
\[ 2y_{53} - y_{45} - y_5 \leq 0 \]
\[ y_{63} - y_{56} - y_6 \geq -1 \]
\[ 2y_{63} - y_{56} - y_6 \leq 0 \]
\[ y_{54} - y_{45} - y_4 \geq -1 \]
\[ 2y_{54} - y_{45} - y_4 \leq 0 \]
\[ y_{64} - y_{56} - y_5 \geq -1 \]
\[ 2y_{64} - y_{56} - y_5 \leq 0 \]
\[ y_{65} - y_{66} - y_6 \geq -1 \]
\[ 2y_{65} - y_{66} - y_6 \leq 0 \]
\[ 100y_2 + 70y_3 + 80y_4 + 60y_5 + 90y_6 \geq 300 \]

\[ Y_j \geq 0, Y_{ij} = 0,1 \]

The optimum solution to the binary IP is given by \( Y_2 = Y_3 = Y_5 = Y_6 = 1, t_2 = 16, t_3 = 36, t_5 = 9, t_6 = 24 \) and \( t_7 = 44 \). The minimum distance travelled is 44. From the \( t_j \) values we observe that the route is 1-5-2-6-3-1 with distance = 44. The total amount collected is 320.

(The solution \( Y_3 = Y_4 = Y_5 = Y_6 = 1 \) gives us a total revenue of 300. This is not chosen because the distance travelled is more).

b) Here the objective function is to Maximize \( 100y_2 + 70y_3 + 80y_4 + 60y_5 + 90y_6 \). The distance constraint is given by \( t_7 \leq 44 \).

The optimum solution is the same as that for case a).

When the distance constraint is \( t_7 \leq 40 \), the optimum solution is \( Y_2 = Y_3 = Y_6 = 1 \) with the tour 1-3-6-2-1 and distance = 40. The revenue collected is 260.

c.) The revenue constraint is now \( 100y_2 + 70y_3 + 80y_4 + 60y_5 + 90y_6 \geq 250 \). The objective function is to Minimize \( t_7 + 2(6 - Y_1 - Y_2 - Y_3 - Y_4 - Y_5 - Y_6) \)

The optimum solution is \( Y_2 = Y_5 = Y_6 = 1 \), The tour is 1-5-2-6-7 with distance = 34. The revenue is 250. Two cities are not visited. The penalty is 4 and the value of the objective function is 38.

Illustration 8.9

Consider a 6 city TSP where the distance data is given in Table 8.4. The salesman wishes to visit as many cities as possible and return to the starting point. There is a restriction that she travels not more than 40 km?

This problem is the same as in Example 8 case b) except that the objective function is to maximize the number of cities covered instead of maximizing the revenue.

The objective function is to Maximize \( y_2 + y_3 + y_4 + y_5 + y_6 \). The distance constraint is given by \( t_7 \leq 40 \).

The optimum solution is \( Y_2 = Y_3 = Y_6 = 1 \) with the tour 1-3-6-2-1 and distance = 40. The maximum number of cities visited is 3.

This problem is exactly the case in Example 8 b) if all the revenues were 1.