Assignment #1 - Solutions

1. Given periodic signal $e^{-\frac{t}{2}}$,$0 \le t \le 2$,with period $T = 2$. Since it is periodic, its power is $\frac{1}{T}\int_0^2 |x(t)|^2 dt = \frac{1}{2}\int_0^2 \left|e^{-\frac{t}{2}}\right|^2 dt = \frac{1}{2}\int_0^2 e^{-t} dt = -\frac{1}{2}\, e^{-t}\Big|_0^2$
   $= \frac{1}{2} \times \left(1 - e^{-2}\right) = \frac{1}{2} \times \left(1 - \frac{1}{e^2}\right)$
   Ans a

2. Given the impulse train $x(t) = \sum_{n=-\infty}^{\infty} \delta\left(t - nT_s\right)$. Observe that this is a series of impulses with one impulse at each nT_s, where n is an integer. Hence, it is periodic with period T_s. The signal $x(t) = \delta(t)$ for $0 \le t < T_s$. Hence, it can be represented as the Discrete Fourier series with fundamental frequency $\frac{1}{T_s}$,
   $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi\frac{kt}{T_s}}$
   c_k can be found as follows
   $c_k = \frac{1}{T_s}\int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} x(t)e^{-j2\pi\frac{kt}{T_s}} dt = \frac{1}{T_s}\int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} \delta(t)e^{-j2\pi\frac{kt}{T_s}} dt = \frac{1}{T_s}$
   Hence, the discrete Fourier series representation is,
   $x(t) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} e^{j2\pi\frac{kt}{T_s}}$
   Ans c

3. As seen from solution above, the discrete Fourier series of $x(t) = \sum_{n=-\infty}^{\infty} \delta\left(t - nT_s\right)$ is $x(t) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} e^{j2\pi\frac{kt}{T_s}}$. The Fourier transform of each $e^{j2\pi\frac{kt}{T_s}}$ is $\delta\left(f - \frac{k}{T_s}\right)$. Hence, the Fourier transform of $x(t) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} e^{j2\pi\frac{kt}{T_s}}$ is $X(f) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T_s}\right)$
   Ans a

4. Consider the sign function $sgn(t) = 1$ if $t > 0$, 0 at $t = 0$ and -1 if $t < 0$. Observe that $\frac{d}{dt} sgn(t) = 2\delta(t)$. Hence,

$$\mathcal{FT}\left(\frac{d}{dt}sgn(t)\right) = 2 \Rightarrow j2\pi f \mathcal{FT}(sgn(t)) = 2 \Rightarrow \mathcal{FT}(sgn(t)) = \frac{1}{j\pi f}$$

We have used the property $j2\pi f X(f)$ is the Fourier transform of $\frac{dx(t)}{dt}$. Further, observe $u(t) = \frac{(1 + sgn(t))}{2}$.

Hence, $U(f) = \frac{1}{2}\mathcal{FT}(1 + sgn(t)) = \frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$

Ans d

5. Notice that

$G(0) = \int\limits_{-\infty}^{\infty} e^{-kt^2} dt$

Also, from the property of the Gaussian pdf $\int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-x^2}{2\sigma^2}} dx = 1$. Hence it follows that

$\int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi \times \frac{1}{2k}}} e^{-kt^2} dt = 1 \Rightarrow \int\limits_{-\infty}^{\infty} e^{-kt^2} dt = \sqrt{2\pi \times \frac{1}{2k}} = \sqrt{\frac{\pi}{k}} = G(0)$

Ans c

6. Also, $G(f)$ can be expressed as, $G(f) = \int\limits_{-\infty}^{\infty} e^{-kt^2} e^{-j2\pi f t} dt$

Differentiating with respect to f,

$$\frac{dG(f)}{df} = \frac{d}{df}\int_{-\infty}^{\infty} e^{-kt^2} e^{-j2\pi f t} dt = \int_{-\infty}^{\infty} e^{-kt^2} \frac{d}{df} e^{-j2\pi f t} dt = \int_{-\infty}^{\infty} e^{-kt^2} (-j2\pi t) e^{-j2\pi f t} dt$$

$$= \frac{(-j2\pi)}{(-2k)} \times \int_{-\infty}^{\infty} (-2kt) e^{-kt^2} e^{-j2\pi f t} dt = \frac{j\pi}{k} \int_{-\infty}^{\infty} \left(\frac{d}{dt} e^{-kt^2}\right) e^{-j2\pi f t} dt = \frac{j\pi}{k} \mathscr{F}\mathscr{T}\left(\frac{d}{dt} e^{-kt^2}\right)$$

$$= \frac{j\pi}{k} \times (j2\pi f) G(f) = -\frac{2\pi^2}{k} f G(f)$$

Hence, $\alpha = \frac{2\pi^2}{k}$

Ans a

7. Solving the differential equation above,

$$\frac{dG(f)}{df} = -\alpha f G(f) \Rightarrow \frac{dG(f)}{G(f)} = -\alpha f \times df \Rightarrow \int_0^f \frac{dG(f')}{G(f')} = -\int_0^f \alpha f' \times df'$$

$$\Rightarrow \ln G(f) - \ln G(0) = -\alpha \frac{f^2}{2} \Rightarrow G(f) = G(0)e^{-\alpha\frac{f^2}{2}} = \sqrt{\frac{\pi}{k}} e^{\frac{2\pi^2}{k}\times\frac{f^2}{2}} = \sqrt{\frac{\pi}{k}} e^{-\frac{\pi^2}{k}f^2}$$

Ans b

8. Given signal $g_p(t)$, it is real and even. Since it is periodic, let its fundamental frequency be f_0. Let its discrete Fourier series be,

$$g_p(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t}$$

$$\Rightarrow g_p^*(t) = \left(\sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t}\right)^* = \sum_{k=-\infty}^{\infty} c_k^* e^{-j2\pi k f_0 t}$$

But since $g_p(t)$ is real, we must have,

$$g_p(t) = g_p^*(t)$$

$$\Rightarrow \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} = \sum_{k=-\infty}^{\infty} c_k^* e^{-j2\pi k f_0 t} = \sum_{k=-\infty}^{\infty} c_{-k}^* e^{j2\pi k f_0 t} \Rightarrow c_k = c_{-k}^*$$

where the last step follows by equating coefficients of $e^{j2\pi k f_0 t}$. Further, given $g_p(t)$ is even

$$g_p(-t) = \sum_{k=-\infty}^{\infty} c_k e^{-j2\pi k f_0 t} = \sum_{k=-\infty}^{\infty} c_{-k} e^{j2\pi k f_0 t} = g_p(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t}$$

$$\Rightarrow c_k = c_{-k}$$

Therefore, we have $c_k = c_{-k}^*$ and $c_k = c_{-k}$, which implies that $c_k = c_k^* = c_{-k}$ i.e. c_k is real and even.

Ans: True

9. $j2\pi f G(f)$ is the Fourier transform of $\frac{dg(t)}{dt}$. Therefore, we must have

$$j2\pi f G(f) = \int_{k=-\infty}^{\infty} \frac{dg(t)}{dt} e^{-j2\pi f t} dt$$

$$\Rightarrow |j2\pi f G(f)| = \left|\int_{-\infty}^{\infty} \frac{dg(t)}{dt} e^{-j2\pi f t} dt\right| \leq \int_{k=-\infty}^{\infty} \left|\frac{dg(t)}{dt} e^{-j2\pi f t}\right| dt = \int_{-\infty}^{\infty} \left|\frac{dg(t)}{dt}\right| dt$$

$$\Rightarrow |j2\pi f G(f)| \leq \int_{-\infty}^{\infty} \left|\frac{dg(t)}{dt}\right| dt$$

Ans: False

10. It can be seen that $\frac{dg(t)}{dt}$ will comprise of two pulses of height 1 from -2 to -1 and 1 to 2. Hence, $\frac{d^2 g(t)}{dt^2}$ will comprise of 4 impulses with scaling 1 at -2, 2 and scaling -1 at -1,1. Hence,

$$\int_{-\infty}^{\infty} \left| \frac{d^2 g(t)}{dt^2} \right| dt = \int_{-\infty}^{\infty} |\delta(t+2) - \delta(t+1) - \delta(t-1) + \delta(t-2)| \, dt = 4$$

Ans d