Module
1
Energy Methods in Structural Analysis

Lesson 5

Virtual Work

Instructional Objectives

After studying this lesson, the student will be able to:

1. Define Virtual Work.
2. Differentiate between external and internal virtual work.
3. Sate principle of virtual displacement and principle of virtual forces.
4. Drive an expression of calculating deflections of structure using unit load method.
5. Calculate deflections of a statically determinate structure using unit load method.
6. State unit displacement method.
7. Calculate stiffness coefficients using unit-displacement method.

5.1 Introduction

In the previous chapters the concept of strain energy and Castigliano's theorems were discussed. From Castigliano's theorem it follows that for the statically determinate structure; the partial derivative of strain energy with respect to external force is equal to the displacement in the direction of that load. In this lesson, the principle of virtual work is discussed. As compared to other methods, virtual work methods are the most direct methods for calculating deflections in statically determinate and indeterminate structures. This principle can be applied to both linear and nonlinear structures. The principle of virtual work as applied to deformable structure is an extension of the virtual work for rigid bodies. This may be stated as: if a rigid body is in equilibrium under the action of a $F-$system of forces and if it continues to remain in equilibrium if the body is given a small (virtual) displacement, then the virtual work done by the $F-$system of forces as 'it rides' along these virtual displacements is zero.

5.2 Principle of Virtual Work

Many problems in structural analysis can be solved by the principle of virtual work. Consider a simply supported beam as shown in Fig.5.1a, which is in equilibrium under the action of real forces $F_1, F_2, ..., F_n$ at co-ordinates $1, 2, ..., n$ respectively. Let $u_1, u_2, ..., u_n$ be the corresponding displacements due to the action of forces $F_1, F_2, ..., F_n$. Also, it produces real internal stresses σ_{ij} and real internal strains ε_{ij} inside the beam. Now, let the beam be subjected to second system of forces (which are virtual not real) $\delta F_1, \delta F_2, ..., \delta F_n$ in equilibrium as shown in Fig.5.1b. The second system of forces is called virtual as they are imaginary and they are not part of the real loading. This produces a displacement configuration $\delta u_1, \delta u_2, \ldots\ldots\ldots, \delta u_n$. The virtual loading system produces virtual internal stresses $\delta\sigma_{ij}$ and virtual internal strains $\delta\varepsilon_{ij}$ inside the beam. Now, apply the second system of forces on the beam which has been deformed by first system of forces. Then, the external loads F_i and internal stresses σ_{ij} do virtual work by moving along δu_i and $\delta\varepsilon_{ij}$. The product $\sum F_i \delta u_i$ is known as the external virtual work. It may be noted that the above product does not represent the conventional work since each component is caused due to different source i.e. δu_i is not due to F_i. Similarly the product $\sum \sigma_{ij} \delta\varepsilon_{ij}$ is the internal virtual work. In the case of deformable body, both external and internal forces do work. Since, the beam is in equilibrium, the external virtual work must be equal to the internal virtual work. Hence, one needs to consider both internal and external virtual work to establish equations of equilibrium.

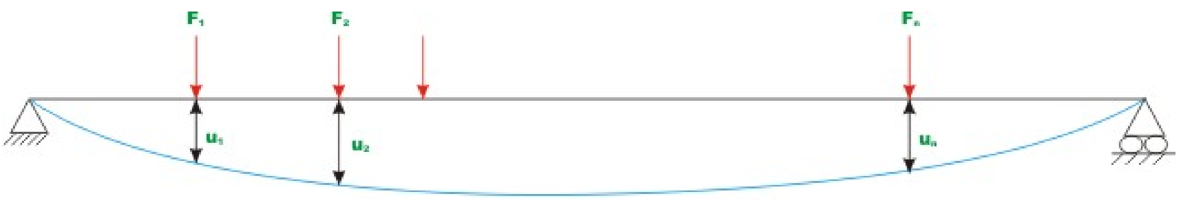

Fig. 5.1a : Actual system of forces.

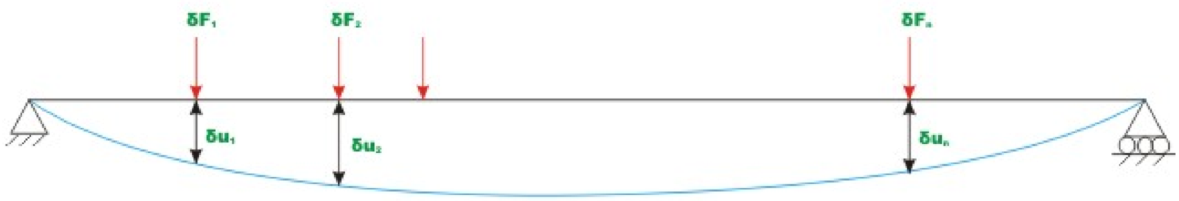

Fig. 5.1b : virtual system of forces.

5.3 Principle of Virtual Displacement

A deformable body is in equilibrium if the total external virtual work done by the system of true forces moving through the corresponding virtual displacements of the system i.e. $\sum F_i \delta u_i$ is equal to the total internal virtual work for every kinematically admissible (consistent with the constraints) virtual displacements.

That is virtual displacements should be continuous within the structure and also it must satisfy boundary conditions.

$$\sum F_i \, \delta u_i = \int \sigma_{ij} \, \delta\varepsilon_{ij} \, dv \tag{5.1}$$

where σ_{ij} are the true stresses due to true forces F_i and $\delta\varepsilon_{ij}$ are the virtual strains due to virtual displacements δu_i.

5.4 Principle of Virtual Forces

For a deformable body, the total external complementary work is equal to the total internal complementary work for every system of virtual forces and stresses that satisfy the equations of equilibrium.

$$\sum \delta F_i \, u_i = \int \delta\sigma_{ij} \, \varepsilon_{ij} \, dv \tag{5.2}$$

where $\delta\sigma_{ij}$ are the virtual stresses due to virtual forces δF_i and ε_{ij} are the true strains due to the true displacements u_i.

As stated earlier, the principle of virtual work may be advantageously used to calculate displacements of structures. In the next section let us see how this can be used to calculate displacements in a beams and frames. In the next lesson, the truss deflections are calculated by the method of virtual work.

5.5 Unit Load Method

The principle of virtual force leads to unit load method. It is assumed throughout our discussion that the method of superposition holds good. For the derivation of unit load method, we consider two systems of loads. In this section, the principle of virtual forces and unit load method are discussed in the context of framed structures. Consider a cantilever beam, which is in equilibrium under the action of a first system of forces $F_1, F_2, \ldots.., F_n$ causing displacements $u_1, u_2, \ldots.., u_n$ as shown in Fig. 5.2a. The first system of forces refers to the actual forces acting on the structure. Let the stress resultants at any section of the beam due to first system of forces be axial force (P), bending moment (M) and shearing force (V). Also the corresponding incremental deformations are axial deformation ($d\Delta$), flexural deformation ($d\theta$) and shearing deformation ($d\lambda$) respectively.

For a conservative system the external work done by the applied forces is equal to the internal strain energy stored. Hence,

$$\frac{1}{2}\sum_{i=1}^{n} F_i u_i = \frac{1}{2}\int P \ \mathrm{d}\Delta + \frac{1}{2}\int M \ \mathrm{d}\theta + \frac{1}{2}\int V \ \mathrm{d}\lambda$$

$$= \int_0^L \frac{P^2 ds}{2EA} + \int_0^L \frac{M^2 ds}{2EI} + \int_0^L \frac{V^2 ds}{2AG} \tag{5.3}$$

Now, consider a second system of forces $\delta F_1, \delta F_2, \ldots\ldots, \delta F_n$, which are virtual and causing virtual displacements $\delta u_1, \delta u_2, \ldots\ldots, \delta u_n$ respectively (see Fig. 5.2b). Let the virtual stress resultants caused by virtual forces be $\delta P_v, \delta M_v$ and δV_v at any cross section of the beam. For this system of forces, we could write

$$\frac{1}{2}\sum_{i=1}^{n} \delta F_i \delta u_i = \int_0^L \frac{\delta P_v^{\ 2} ds}{2EA} + \int_0^L \frac{\delta M_v^{\ 2} ds}{2EI} + \int_0^L \frac{\delta V_v^{\ 2} ds}{2AG} \tag{5.4}$$

where $\delta P_v, \delta M_v$ and δV_v are the virtual axial force, bending moment and shear force respectively. In the third case, apply the first system of forces on the beam, which has been deformed, by second system of forces $\delta F_1, \delta F_2, \ldots\ldots, \delta F_n$ as shown in Fig 5.2c. From the principle of superposition, now the deflections will be $(u_1 + \delta u_1), (u_2 + \delta u_2), \ldots\ldots, (u_n + \delta u_n)$ respectively

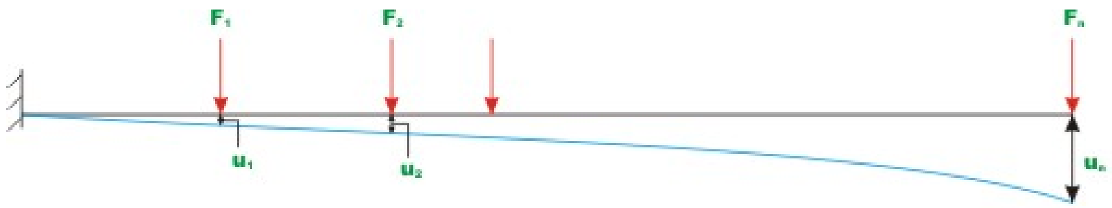

Fig. 5.2a : Actual system.

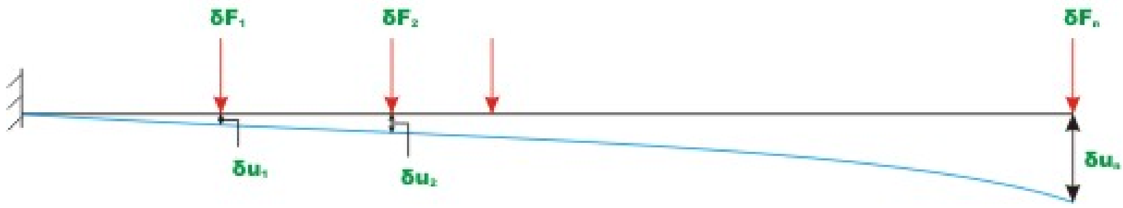

Fig. 5.2b : Virtual system of forces.

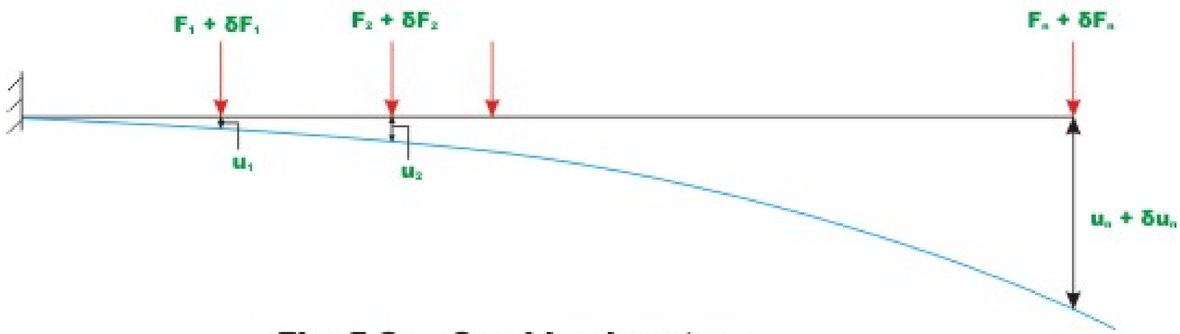

Fig. 5.2c : Combined system.

Since the energy is conserved we could write,

$$\frac{1}{2}\sum_{j=1}^{n} F_j u_j + \frac{1}{2}\sum_{j=1}^{n} \delta F_j \delta u_j + \sum_{j=1}^{n} \delta F_j u_j = \int_0^L \frac{\delta P_v^2 ds}{2EA} + \int_0^L \frac{\delta M_v^2 ds}{2EI} + \int_0^L \frac{\delta V_v^2 ds}{2AG} + \int_0^L \frac{P^2 ds}{2EA} + \int_0^L \frac{M^2 ds}{2EI} + \int_0^L \frac{V^2 ds}{2AG} + \int_0^L \delta P_v d\Delta + \int_0^L \delta M_v d\theta + \int_0^L \delta V_v d\lambda \qquad (5.5)$$

In equation (5.5), the term on the left hand side $\left(\sum \delta F_j u_j\right)$, represents the work done by virtual forces moving through real displacements. Since virtual forces act at its full value, $\left(\frac{1}{2}\right)$ does not appear in the equation. Subtracting equation (5.3) and (5.4) from equation (5.5) we get,

$$\sum_{j=1}^{n} \delta F_j u_j = \int_0^L \delta P_v d\Delta + \int_0^L \delta M_v d\theta + \int_0^L \delta V_v d\lambda \tag{5.6}$$

From Module 1, lesson 3, we know that

$d\Delta = \frac{Pds}{EA}, d\theta = \frac{Mds}{EI}$ and $d\lambda = \frac{Vds}{AG}$. Hence,

$$\sum_{j=1}^{n} \delta F_j u_j = \int_0^L \frac{\delta P_v Pds}{EA} + \int_0^L \frac{\delta M_v Mds}{EI} + \int_0^L \frac{\delta V_v Vds}{AG} \tag{5.7}$$

Note that $\left(\frac{1}{2}\right)$ does not appear on right side of equation (5.7) as the virtual system resultants act at constant values during the real displacements. In the present case $\delta P_v = 0$ and if we neglect shear forces then we could write equation (5.7) as

$$\sum_{j=1}^{n} \delta F_j u_j = \int_0^L \frac{\delta M_v Mds}{EI} \tag{5.8}$$

If the value of a particular displacement is required, then choose the corresponding force $\delta F_i = 1$ and all other forces $\delta F_j = 0 \quad \left(j = 1,2,....,i-1,i+1,....,n\right)$. Then the above expression may be written as,

$$(1)u_i = \int_0^L \frac{\delta M_v Mds}{EI} \tag{5.9}$$

where δM_v are the internal virtual moment resultants corresponding to virtual force at *i*-th co-ordinate, $\delta F_i = 1$. The above equation may be stated as,

$$(unit \quad virtual \quad load\ )\ \ unknown \quad true\ displacement = \int\left(virtual \quad stress \quad resultants\right)\left(real \quad deformations\right)\ ds. \tag{5.10}$$

The equation (5.9) is known as the unit load method. Here the unit virtual load is applied at a point where the displacement is required to be evaluated. The unit load method is extensively used in the calculation of deflection of beams, frames and trusses. Theoretically this method can be used to calculate deflections in statically determinate and indeterminate structures. However it is extensively used in evaluation of deflections of statically determinate structures only as the method requires a priori knowledge of internal stress resultants.

Example 5.1

A cantilever beam of span L is subjected to a tip moment M_0 as shown in Fig 5.3a. Evaluate slope and deflection at a point $\left(\frac{3L}{4}\right)$ from left support. Assume EI of the given beam to be constant.

Fig. 5.3a Example 5.1

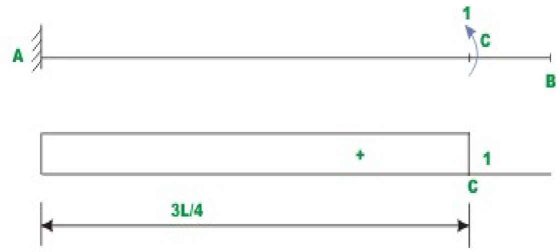

Fig. 5.3c. B. M. diagram of the beam due to unit moment at C.

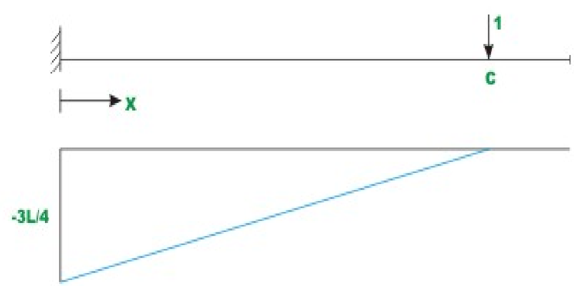

Fig. 5.3d B.M.D due to unit load at C

Fig. 5.3b : B. M. diagram of the beam due to moment M_0.

Slope at C

To evaluate slope at C, a virtual unit moment is applied at C as shown in Fig 5.3c. The bending moment diagrams are drawn for tip moment M_0 and unit moment applied at C and is shown in fig 5.3b and 5.3c respectively. Let θ_c be the rotation at C due to moment M_0 applied at tip. According to unit load method, the rotation at C, θ_c is calculated as,

$$(1)\theta_c = \int_0^L \frac{\delta M_v(x) M(x) dx}{EI} \quad (1)$$

where $\delta M_v(x)$ and $M(x)$ are the virtual moment resultant and real moment resultant at any section x. Substituting the value of $\delta M_v(x)$ and $M(x)$ in the above expression, we get

$$(1)\theta_c = \int_0^{3L/4} \frac{(1)Mdx}{EI} + \int_{3L/4}^{L} \frac{(0)Mdx}{EI}$$

$$\theta_c = \frac{3ML}{4EI} \quad (2)$$

Vertical deflection at C

To evaluate vertical deflection at C, a unit virtual vertical force is applied ac C as shown in Fig 5.3d and the bending moment is also shown in the diagram. According to unit load method,

$$(1)u_A = \int_0^L \frac{\delta M_v(x) M(x) dx}{EI} \quad (3)$$

In the present case, $\delta M_v(x) = -\left(\frac{3L}{4} - x\right)$

and $M(x) = +M_0$

$$u_A = \int_0^{\frac{3L}{4}} \frac{-\left(\frac{3L}{4} - x\right)M}{EI} dx$$

$$= -\frac{M}{EI} \int_0^{\frac{3L}{4}} \left(\frac{3L}{4} - x\right) dx$$

$$= -\frac{M}{EI} \left[\frac{3L}{4}x - \frac{x^2}{2}\right]_0^{\frac{3L}{4}}$$

$$= -\frac{9ML^2}{32EI} (\uparrow) \quad (4)$$

Example 5.2

Find the horizontal displacement at joint B of the frame ABCD as shown in Fig. 5.4a by unit load method. Assume EI to be constant for all members.

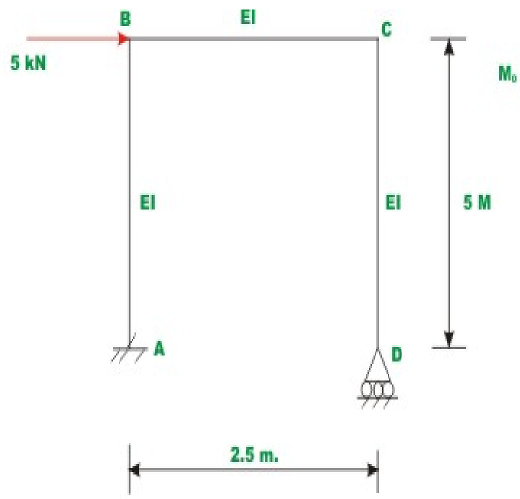

Fig. 5.4 a Example 5.2

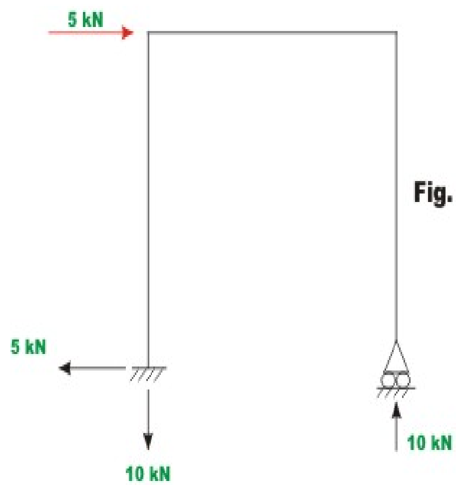

Fig. 5.4 b. Reactions.

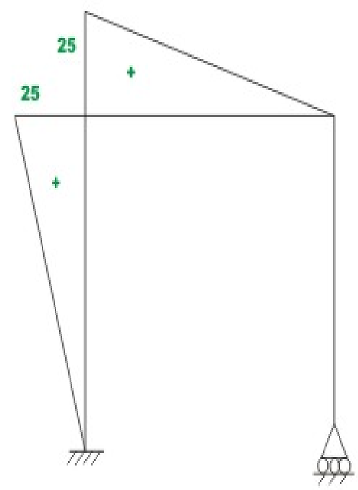

Fig. 5.4 c. Bending moment diagram of the frame for external loading.

The reactions and bending moment diagram of the frame due to applied external loading are shown in Fig 5.4b and Fig 5.4c respectively. Since, it is required to calculate horizontal deflection at B, apply a unit virtual load at B as shown in Fig. 5.4d. The resulting reactions and bending moment diagrams of the frame are shown in Fig 5.4d.

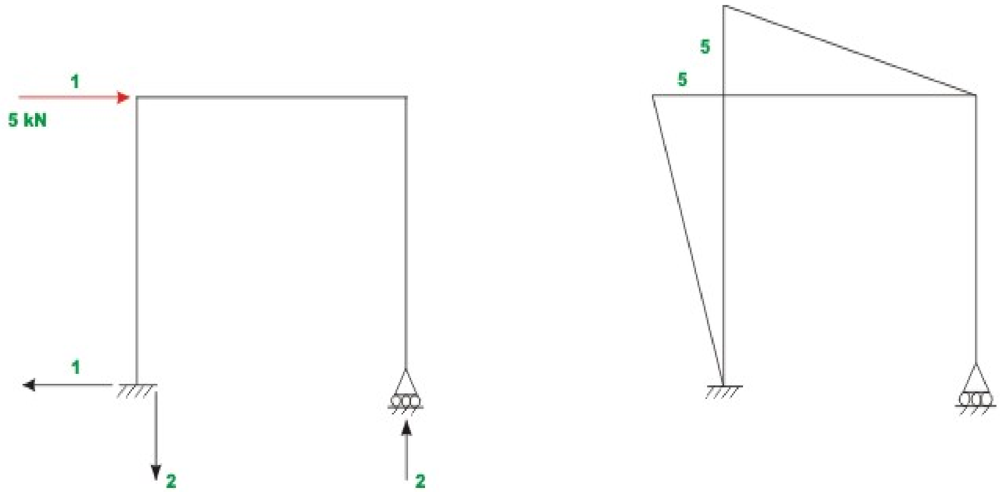

Fig. 5.4 d. Reactions and bending moment diagram of the frame for unit vertical load applied at B.

Now horizontal deflection at B, u_B may be calculated as

$$(1)\times u_H^B = \int_A^D \frac{\delta M_v(x) M(x) dx}{EI} \quad (1)$$

$$= \int_A^B \frac{\delta M_v(x) M(x) dx}{EI} + \int_B^C \frac{\delta M_v(x) M(x) dx}{EI} + \int_C^D \frac{\delta M_v(x) M(x) dx}{EI}$$

$$= \int_0^5 \frac{(x)(5x)dx}{EI} + \int_0^{2.5} \frac{2(2.5-x)10(2.5-x)dx}{EI} + 0$$

$$= \int_0^5 \frac{(5x^2)dx}{EI} + \int_0^{2.5} \frac{20(2.5-x)^2 dx}{EI}$$

$$= \frac{625}{3EI} + \frac{312.5}{3EI} = \frac{937.5}{3EI}$$

Hence, $u_A = \dfrac{937.5}{3EI} (\rightarrow)$ (2)

Example 5.3

Find the rotations of joint B and C of the frame shown in Fig. 5.4a. Assume EI to be constant for all members.

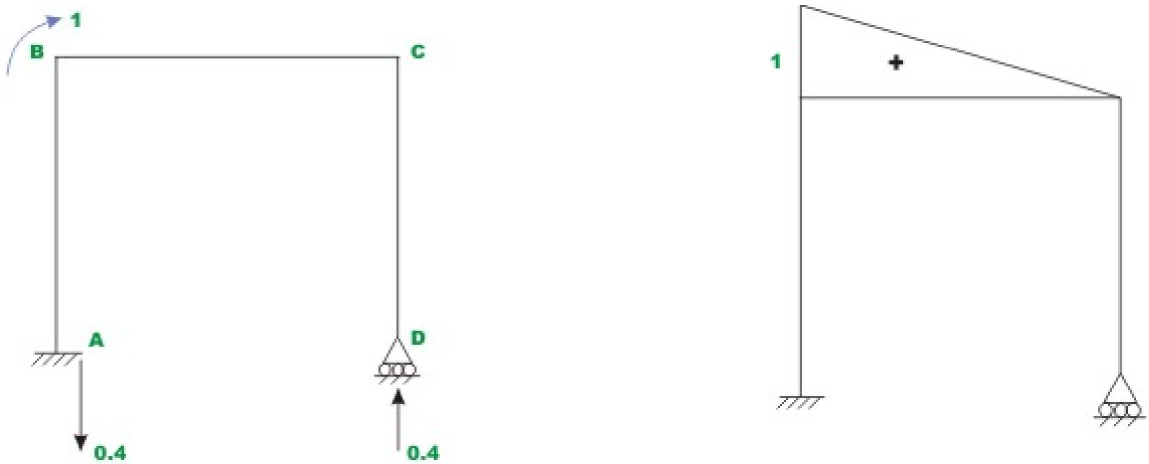

Fig. 5.5a. Reaction and B. M. diagram for the unit moment applied at B.

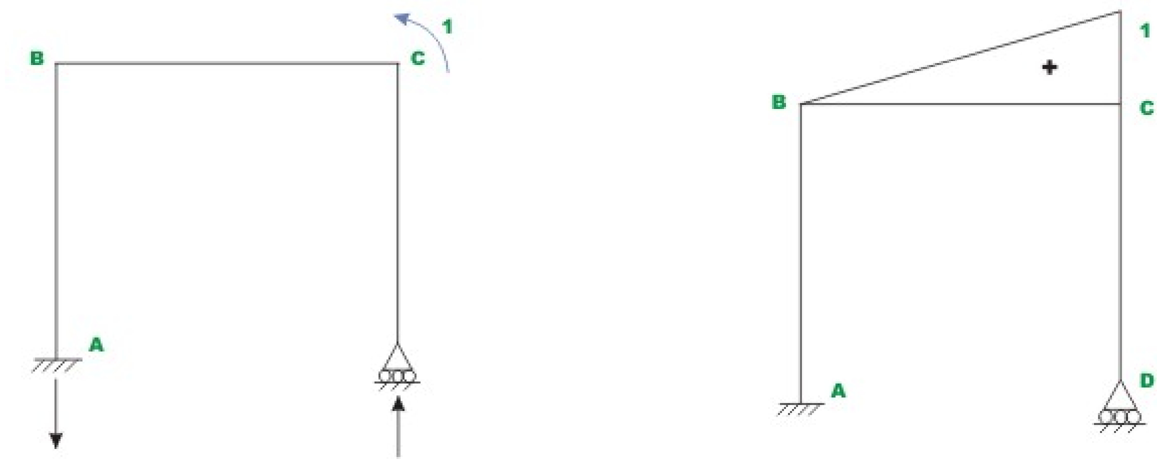

Fig. 5.5b. Reaction and B. M. diagram for the unit moment applied at C.

Rotation at B

Apply unit virtual moment at B as shown in Fig 5.5a. The resulting bending moment diagram is also shown in the same diagram. For the unit load method, the relevant equation is,

$$(1)\times\theta_B = \int_A^D \frac{\delta M_v(x) M(x) dx}{EI} \qquad (1)$$

wherein, θ_B is the actual rotation at B, $\delta M_v(x)$ is the virtual stress resultant in the frame due to the virtual load and $\int_A^D \frac{M(x)}{EI}dx$ is the actual deformation of the frame due to real forces.

Now, $M(x)=10(2.5-x)$ and $\delta M_v(x)=0.4(2.5-x)$
Substituting the values of $M(x)$ and $\delta M_v(x)$ in the equation (1),

$$\theta_B=\frac{4}{EI}\int_0^{2.5}(2.5-x)^2\,dx$$

$$=\frac{4}{EI}\left[6.25x-\frac{5x^2}{2}+\frac{x^3}{3}\right]_0^{2.5}=\frac{62.5}{3EI} \qquad (2)$$

Rotation at C

For evaluating rotation at C by unit load method, apply unit virtual moment at C as shown in Fig 5.5b. Hence,

$$(1)\times\theta_C=\int_A^D\frac{\delta M_v(x)M(x)dx}{EI} \qquad (3)$$

$$\theta_C=\int_0^{2.5}\frac{10(2.5-x)(0.4x)}{EI}dx$$

$$=\frac{4}{EI}\left[\frac{2.5x^2}{2}-\frac{x^3}{3}\right]_0^{2.5}=\frac{31.25}{3EI} \qquad (4)$$

5.6 Unit Displacement Method

Consider a cantilever beam, which is in equilibrium under the action of a system of forces $F_1,F_2,.....,F_n$. Let $u_1,u_2,.....,u_n$ be the corresponding displacements and P,M and V be the stress resultants at section of the beam. Consider a second system of forces (virtual) $\delta F_1,\delta F_2,.....,\delta F_n$ causing virtual displacements $\delta u_1,\delta u_2,.....,\delta u_n$. Let $\delta P_v,\delta M_v$ and δV_v be the virtual axial force, bending moment and shear force respectively at any section of the beam.
Apply the first system of forces $F_1,F_2,.....,F_n$ on the beam, which has been previously bent by virtual forces $\delta F_1,\delta F_2,.....,\delta F_n$. From the principle of virtual displacements we have,

$$\sum_{j=1}^{n}F_j\delta u_j=\int\frac{M(x)\delta M_v(x)ds}{EI}$$

$$=\int_V\sigma^T\delta\varepsilon\,\delta v \qquad (5.11)$$

The left hand side of equation (5.11) refers to the external virtual work done by the system of true/real forces moving through the corresponding virtual displacements of the system. The right hand side of equation (5.8) refers to internal virtual work done. The principle of virtual displacement states that the external virtual work of the real forces multiplied by virtual displacement is equal to the real stresses multiplied by virtual strains integrated over volume. If the value of a particular force element is required then choose corresponding virtual displacement as unity. Let us say, it is required to evaluate F_1, then choose $\delta u_1 = 1$ and $\delta u_i = 0 \quad i = 2,3,.....,n$. From equation (5.11), one could write,

$$(1)\ F_1 = \int \frac{M\,(\delta M_v)_1\,ds}{EI} \tag{5.12}$$

where, $(\delta M_v)_1$ is the internal virtual stress resultant for $\delta u_1 = 1$. Transposing the above equation, we get

$$F_1 = \int \frac{(\delta M_v)_1 M ds}{EI} \tag{5.13}$$

The above equation is the statement of unit displacement method. The above equation is more commonly used in the evaluation of stiffness co-efficient k_{ij} . Apply real displacements $u_1,.....,u_n$ in the structure. In that set $u_2 = 1$ and the other all displacements $u_i = 0 \quad (i = 1,3,......,n)$. For such a case the quantity F_j in equation (5.11) becomes k_{ij} *i.e.* force at 1 due to displacement at 2. Apply virtual displacement $\delta u_1 = 1$. Now according to unit displacement method,

$$(1)\ k_{12} = \int \frac{(\delta M_v)_1 M_2 ds}{EI} \tag{5.14}$$

Summary

In this chapter the concept of virtual work is introduced and the principle of virtual work is discussed. The terms internal virtual work and external virtual work has been explained and relevant expressions are also derived. Principle of virtual forces has been stated. It has been shown how the principle of virtual load leads to unit load method. An expression for calculating deflections at any point of a structure (both statically determinate and indeterminate structure) is derived. Few problems have been solved to show the application of unit load method for calculating deflections in a structure.